Post-Riemannian Merger of Yang-Mills Interactions with Gravity

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Abstract

We show that a post-Riemannian spacetime can accommodate an internal symmetry structure of the Yang-Mills prototype in such a way that the internal symmetry becomes an integral part of the spacetime itself. The construction encrusts the internal degrees of freedom in spacetime in a manner that merges the gauging of these degrees of freedom with the frame geometrical gauges of spacetime. In particular, we prove that the three spacetime structural identities, which now become "contaminated" by internal degrees of freedom, remain invariant with respect to internal gauge transformations. In a Weyl Cartan spacetime, the theory regains the original form of Einstein's equations, in which gauge field sources on the r.h.s. determine on the l.h.s the geometry of spacetime and the fields it induces. In the more general case we identify new contributions of weak magnitude in the interaction between the Yang-Mills field and gravity. The merger of spacetime with internal degrees of freedom which we propose here is not constrained by the usual Coleman-Mandula considerations.

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1 Introduction

In this paper we prove an equivalence theorem between, on the one hand, Yang-Mills field, gauging an internal Abelian or non-Abelian symmetry group U, *in the presence of Einstein gravity*, and on the other hand, a regrouping in which Einstein's gravity *and* the internal symmetry are replaced by an *extended post-Riemannian spacetime theory* of the Weyl-Cartan type, which includes torsion and Weyl's non-metricity. The theory thereby regains the original structure of Einstein's equations, in which gauge field sources on the r.h.s. determine on the l.h.s the geometry of spacetime and the fields it induces.

The construction *encrusts* the internal degrees of freedom in spacetime in such a way as to *merge* the gauging of these internal degrees of freedom with the frame geometrical gauges of spacetime. Note that *unification* of gravity with gauge theories, such as those of the Standard Model, has up to now been achieved either through supersymmetry (as in supergravity) or with new spatial dimensions (as in Kaluza-Klein methodology). Note that putting together the Gravity and Yang-Mills connection, i.e. producing the two from the same fabric - this does occur in String Theory, where both come as *excitations of strings*, either closed or open.

This equivalence theorem is extended in a straightforward manner, so that the resulting merged solution be made to correspond to more general post-Riemannian configurations and their corresponding spacetime theory. However, whereas the Weyl-Cartan regrouping may be regarded as no more than a formal (but by no means trivial) geometrical pasting of gauge symmetries, here the presence of Abelian components in the internal gauge group gives rise to new contributions of strength weaker than the gravitational magnitude in the interaction between the Yang-Mills field and gravity.

Note that this *merger* of spacetime with internal degrees of freedom is not constrained by

the usual Coleman-Mandula considerations. The physical states in the Hilbert space of the gauge particles are unmodified and can anyhow only be presented in the original input picture of the Yang-Mills field (and this only in quasi-flat background, i.e. if it can be assumed that it occurs in the presence of weak gravity) as we have no particle Hilbert space description containing information about the geometry of spacetime.

The structure and the content of this paper go as follows:

Section 2 is a pedagogical survey served to fix the language, and to define the framework and tools we shall later intensively utilize. A brief (but comprehensive) analysis of the structure of spacetime with torsion and non-metricity is given, with an emphasis on the gauge principle underlying the theory. Two auxiliary constructions which improve the derivations have been included in this prelude:

1. Replacing the conventional "*reductive*" formalism for post-Riemannian geometry by a "*constructive*" approach. Explicitly, the reductive approach consists in defining the connection and frames for an-holonomic $GL(n, \mathbb{R})$, then splitting the geometry into its Riemannian piece with the Christoffel connection gauging the orthogonal subgroup, the Weyl component gauging dilations, and the traceless non-metricity component gauging the shears. In the constructive systematics, the connection and the frames are originally introduced in the Riemannian framework, with the connection then undergoing a *deformation*. The post-Riemannian structure then emerges as a deformation of the original Riemannian structure.
2. Extensive use of absolute differentials.

section 3 presents a formalism in which internal symmetry is naturally embedded *within* the non-metric part of spacetime such that the internal symmetry becomes an integral

part of the spacetime itself. Namely, the basespace components of the gauge field of the embedded symmetry, *dictated* to be uniquely of the Yang-Mills prototype, are unified with those of the spacetime connection. Thereby, our merging theory relies on a *single* connection 1-form; we shall elaborate on its non-usual algebraic structure, and on the means by which the metric sector remains protected against invasions from the internal world. We shall explicitly re-derive the relevant spacetime structural identities (now containing internal degrees of freedom as well), and prove that they are invariant with respect to internal gauge transformations.

Section 4 deals with dynamics. Starting from a minimal variant of the Einstein-Hilbert action, suitable for a post-Riemannian spacetime, we analyze in details the dynamical structure of the Weyl-Cartan merger. In particular, we re-derive Einstein's equation in terms of a fancy equality between the Einstein $(n-1)$-form and the energy-momentum current composed of the Yang-Mills field strength. The latter object is shown to be traceless only in four dimensions, in which case self dual solutions to the Yang-Mills equations are seen to correspond to *non-trivial gravitational emptiness*. Working in a preferred gauge, we turn to study the more general case in a qualitative manner. The gauge fixing process endows the gauge fields with a scaling function, by thus making them sensitive to the non-metric microstructure of spacetime. We isolate those terms in the action in which these fields interact with the gravitational field, and find that the magnitude of these interactions is weaker than the gravitational magnitude.

Section 5 contains summarizing and closing remarks.

We have also included in this work a ***supplement*** where we develop some integration formulas, and discuss some of the topological aspects of the theory. Among other things,

we calculate the charge of the 1-st Chern class associated with the merger, and formulate analytically two *non-Abelian* generalizations to Stokes' theorem. The first of them concerns the Weyl covector (a formula to which we also refer in the text); the second concerns the coframe-torsion transvection field.

Some technical conventions:

In what follows we shall extensively employ Cartan's differential calculus, reinforced by the powerful concept of the *absolute differential* [1] which is a generalized exterior derivative: it operates as an ordinary exterior derivative on forms, but when it is applied to objects such as vectors and tensors, it generates (here $O(n)$-) covariant exterior derivatives. A rough (and rather intuitive) definition will be given below.

Consider a tensor p-form $\boldsymbol{t}_p$ which we may schematically write as $\boldsymbol{t}_p = t[\vartheta][\boldsymbol{e}]$. Here t denotes its components (or coordinates), $[\vartheta]$ the Grassmann basis, and $[\boldsymbol{e}]$ the tensorial basis. The 1-st and 2-nd order absolute differentials of $\boldsymbol{t}_p$ are given by:

$$
\begin{aligned}
d\boldsymbol{t}_p &= d(t[\vartheta])[\boldsymbol{e}] + (-1)^p t[\vartheta] \wedge d[\boldsymbol{e}], \\
dd\boldsymbol{t}_p &= t[\vartheta] \wedge dd[\boldsymbol{e}];
\end{aligned}
\tag{1}
$$

note that in our paradigm d^2 fails to annihilate on tensor bases. Instead, as with covariant exterior derivatives, it will be shown to generate the $O(n)$-curvature. But for scalar-valued forms we have: $d(t[\vartheta]) = dt \wedge [\vartheta] + td[\vartheta]$, and clearly $d^2(t[\vartheta]) = 0$.

Round brackets around a cluster of indices of an object, as in $Q_{\alpha(\beta\gamma)}$, designate complete symmetrization; square brackets instead, as in $R_{[\alpha\beta]}$, designate complete anti-symmetrization (no factorial terms are included: $R_{[\alpha\beta]} := R_{\alpha\beta} - R_{\beta\alpha}$ etc.). Bars added in between indices arranged in a cluster, such as in $\varpi_{[\alpha|\beta|\gamma]}$, tell which indices are to be excluded in the sym-

metrization process (here β). We shall often employ graded brackets,

$$[\![\alpha, \beta]\!] := \alpha \wedge \beta - (-1)^{(\deg\alpha)(\deg\beta)}\, \beta \wedge \alpha, \tag{2}$$

and only in few occasions refer to the ordinary commutator, $[\alpha, \beta] = \alpha \wedge \beta - \beta \wedge \alpha$.

The substratum in our model is an n dimensional differentiable manifold which we shall simply denote by Ω (the events continuum). The signature it carries is (p, q) $(p + q = n)$ but our formalism, in general, is not sensitive to the signature. We shall sometimes refer to an arbitrary m-domain $(m \leq n)$, in which case we shall employ the letter Σ instead. Finally, we use the notation $T\Omega^{\otimes k}$ to denote the bundle of product space fibers, each of whose fibers is a space product of k copies of the tangent space (or the bundle of rank-k tensors over Ω).

2 A spacetime with torsion and non-metricity

2.1 Employing the bundle of tangent frames by means of absolute differentials

The building blocks in the gauge theory of spacetime are the *frame fields*. Until otherwise stated, holonomic frame elements attach a Greek index, $\boldsymbol{e}_\alpha$, whereas non-holonomic frames attach a Latin index, $\boldsymbol{e}_a$; note: the $\{\boldsymbol{e}_a\}$ system is not a-priori constrained to be rectilinear. The elements of the metric tensor in the holonomic and the an-holonomic bases are given, respectively, by $g_{\alpha\beta} := \boldsymbol{e}_\alpha \cdot \boldsymbol{e}_\beta$ and $\bar{g}_{ab} := \boldsymbol{e}_a \cdot \boldsymbol{e}_b$, by means of which the scalar product is defined. The inverse metric tensor is assumed to exist, and its elements with superscript indices are defined through the relations $g_{\alpha\beta} g^{\beta\gamma} = \delta^\gamma_\alpha$, and $\bar{g}_{ab}\bar{g}^{bc} = \delta^c_a$.

Substituting $\boldsymbol{e}^\alpha := \boldsymbol{e}_\beta g^{\beta\alpha}$, and $\boldsymbol{e}^a := \boldsymbol{e}_b \bar{g}^{ba}$ for *inverse frames*, leads to the orthonormality relations $\boldsymbol{e}_\alpha \cdot \boldsymbol{e}^\beta = \delta^\beta_\alpha$, and $\boldsymbol{e}_a \cdot \boldsymbol{e}^b = \delta^b_a$, whence $\boldsymbol{e}_\alpha \cdot \boldsymbol{e}^\alpha = \boldsymbol{e}_a \cdot \boldsymbol{e}^a = n$. Notice: the inverse frame shouldn't be confused with the concept of a *coframe* discussed later in section 2.4. The

n-bein e^a_α in the linear expansion $\boldsymbol{e}_\alpha = e^a_\alpha \boldsymbol{e}_a$, constitutes the *coordinate vector* of $\boldsymbol{e}_\alpha$ in the basis $\{\boldsymbol{e}_a\}$ at x. Consequently, the relation $g_{\alpha\beta} = e^a_\alpha e^b_\beta \bar{g}_{ab}$ is frequently interpreted as a dot product between two coordinate vectors with x-dependent mediating metric tensor $\bar{\boldsymbol{g}}$. From $\boldsymbol{e}_\alpha = e^a_\alpha \boldsymbol{e}_a =: e^a_\alpha \bar{e}^\beta_a \boldsymbol{e}_\beta$ one deduces that $e^a_\alpha \bar{e}^\beta_a = \delta^\beta_\alpha$, and similarly, from $\boldsymbol{e}_a = \bar{e}^\alpha_a \boldsymbol{e}_\alpha = \bar{e}^\alpha_a e^b_\alpha \boldsymbol{e}_b$ one has: $e^a_\alpha \bar{e}^\alpha_b = \delta^a_b$.

A *metric-compatible* connection ω can locally be defined through the *absolute differential* of a frame field.[1] In a holonomic basis,

$$\lim_{\Delta x \to 0} \left[\boldsymbol{e}_\alpha \left(x + \Delta x\right)\big|_x - \boldsymbol{e}_\alpha \left(x\right)\big|_x \right] =: -\omega^\beta{}_\alpha \left(x\right) \boldsymbol{e}_\beta \left(x\right)\big|_x ; \tag{3}$$

namely, we expand the difference between the value of a given frame field at $x + \Delta x$ (basis vector for the tangent space at $x + \Delta x$), displaced in a "parallel" manner to the tangent space at x where we take its measure, and the value of that same frame field at x (basis vector for the tangent space at x), in terms of the frame system at x, as the distance Δx on Ω between the two points (associated with these two tangent spaces) approaches zero.[2]

Eq. (3) may also take the abbreviated form:[3]

$$\boldsymbol{d}\boldsymbol{e}_\alpha = -\omega^\beta{}_\alpha \boldsymbol{e}_\beta \quad \overset{\text{def}}{\Rightarrow} \quad \left(D_\omega \boldsymbol{e}\right)_\alpha = 0; \tag{4}$$

the frames are then said to be covariant-free, or *parallel*. But: the value of the vector 1-form $\boldsymbol{d}\boldsymbol{e}_\alpha$ at x depends on how the basis vector $\boldsymbol{e}_\alpha \left(x + \Delta x\right)$ was transported from the tangent space at $x + \Delta x$ to the tangent space at x. The displacement method is reflected in the form of the expansion coefficients, and it is only in this sense that the frames are regarded as parallel. In an obvious manner, in a non-holonomic basis, we have: $\boldsymbol{d}\boldsymbol{e}_a = -\bar{\omega}^b{}_a \boldsymbol{e}_b \;\; \Rightarrow \;\; \left(D_{\bar{\omega}} \boldsymbol{e}\right)_a = 0$.

[1] The concept of the absolute differential is rigorously presented in [1] where it is also extensively used.

[2] If applied to *functions* on Ω, the limiting procedure in the l.h.s. of (3) generates ordinary differentials.

[3] From $d\left(\boldsymbol{e}_\alpha \cdot \boldsymbol{e}^\beta\right) = 0$, the absolute differential of an inverse frame must come with $+\omega$, $\boldsymbol{d}\boldsymbol{e}^\alpha = \omega^\alpha{}_\beta \boldsymbol{e}^\beta$.

The invariance of definition (4) (whatever the basis one deals with) with respect to linear transformations,

$$\boldsymbol{e} \mapsto \ell \boldsymbol{e}, \quad \text{where} \quad \ell \in L\left(n, \mathbb{R}\right), \; \text{ the frame transformation group,} \tag{5}$$

can only be guaranteed if the 1-form ω transforms as

$$\omega \quad \mapsto \quad \ell \omega \ell^{-1} + \ell d \ell^{-1}. \tag{6}$$

This establishes that ω indeed serves as a connection for the frame bundle. Clearly, the linear transformations applied to an-holonomic frames may form a group as large as $GL\left(n, \mathbb{R}\right)$. But in the approach promoted here, the concept of metric-compatibility (of the connection) will originate from the covariant closure of the frames in all bases (*regardless* of the form of $\bar{g}$), and the loss of this compatibility will be associated with a connection deformation process.

Consider a vector field in its holonomic form, $\boldsymbol{v} = v^\alpha \boldsymbol{e}_\alpha$. This is obviously an invariant object. Applying definition (3), the "exterior derivative" of $\boldsymbol{v}$ reads:

$$d\boldsymbol{v} = \left(dv^\alpha - v^\beta \omega^\alpha{}_\beta \right) \boldsymbol{e}_\alpha = \left(D_\omega v \right)^\alpha \boldsymbol{e}_\alpha. \tag{7}$$

By formulas (5)-(6), $d\boldsymbol{v}$ - the *absolute differential of* $\boldsymbol{v}$ - is a vector 1-form whose coefficients are given by $\left(D_\omega v \right)^\alpha$. More generally, the components of the covariant exterior derivative of a tensor p-form $\boldsymbol{t}_p$ constitute the *coordinate tensor* of its absolute differential:

$$\begin{aligned} d\boldsymbol{t}_p &= d\left(t_p{}^{\alpha_1 \alpha_2 \ldots}{}_{\beta_1 \beta_2 \ldots} \boldsymbol{e}_{\alpha_1 \alpha_2 \ldots}{}^{\beta_1 \beta_2 \ldots} \right) \\ &= \Big(dt_p{}^{\alpha_1 \alpha_2 \ldots}{}_{\beta_1 \beta_2 \ldots} + (-1)^{p+1} t_p{}^{\gamma \alpha_2 \ldots}{}_{\beta_1 \beta_2 \ldots} \wedge \omega^{\alpha_1}_{\gamma} + (-1)^{p+1} t_p{}^{\alpha_1 \gamma \ldots}{}_{\beta_1 \beta_2 \ldots} \wedge \omega^{\alpha_2}_{\gamma} + \cdots \\ &\quad + (-1)^p t_p{}^{\alpha_1 \alpha_2 \ldots}{}_{\gamma \beta_2 \ldots} \wedge \omega^\gamma{}_{\beta_1} + (-1)^p t_p{}^{\alpha_1 \alpha_2 \ldots}{}_{\beta_1 \gamma \ldots} \wedge \omega^\gamma{}_{\beta_2} + \cdots \Big) \boldsymbol{e}_{\alpha_1 \alpha_2 \ldots}{}^{\beta_1 \beta_2 \ldots} \\ &=: \left(D_\omega t_p \right)^{\alpha_1 \alpha_2 \ldots}{}_{\beta_1 \beta_2 \ldots} \boldsymbol{e}_{\alpha_1 \alpha_2 \ldots}{}^{\beta_1 \beta_2 \ldots} \; =: \; \left(D_\omega \boldsymbol{t} \right)_{p+1}. \end{aligned} \tag{8}$$

When people often speak of objects such as vectors or tensors, they usually address, respectively, to coordinate vectors and coordinate tensors (with respect to a given basis). Here we

shall distinguish between vectors ($\boldsymbol{v}$), p-forms (h_p), and tensor p-forms ($\boldsymbol{t}_p$), all of which are different types of *invariants*, and between coordinate vectors (v^α), coordinate tensor p-forms ($t_p^{\alpha_1\alpha_2\ldots}{}_{\beta_1\beta_2\ldots}$), frame elements ($\boldsymbol{e}_\alpha$), and coframes ($\vartheta^\alpha$), all of which transform in a *covariant* manner. Note that, whereas v^α is *scalar-valued*, $\boldsymbol{e}_\alpha$ is *vector-valued*.

Let us turn back again to the transition functions between bases. Hitting a transition function $e_\alpha^a = \boldsymbol{e}_\alpha \cdot \boldsymbol{e}^a$ by an exterior derivative yields

$$de_\alpha^a = -\omega^\beta{}_\alpha e_\beta^a + \bar{\omega}^a{}_b e_\alpha^b, \tag{9}$$

which can be rewritten as $(D_{\omega\bar{\omega}}e)_\alpha^a = 0$. In this respect, the transition functions are regarded as the components of a covariant-free section in the spliced bundle $T\Omega^{\otimes(1,1)}$, each of whose fiberspaces is a space product of two tangent spaces, one employs holonomic basis, the other employs an-holonomic basis, and there is a distinct connection 1-form for each factor fiber in the splice. For this reason we prefer to mark the connection in a non-holonomic basis with a bar, and by thus distinguish it from a connection in a holonomic basis.

Multiplying both sides of eq. (9) by $\bar{e}_a^\gamma$ enables to eliminate ω; otherwise, multiplying it by $\bar{e}_c^\alpha$ enables to eliminate $\bar{\omega}$:

$$\omega^\gamma{}_\alpha = \bar{e}_a^\gamma \bar{\omega}^a{}_b e_\alpha^b - \bar{e}_a^\gamma de_\alpha^a\,; \qquad \bar{\omega}^a{}_c = \bar{e}_c^\alpha \omega^\beta{}_\alpha e_\beta^a + \bar{e}_c^\alpha de_\alpha^a. \tag{10}$$

The connections in the two bases are therefore interrelated through a $GL\,(n,\mathbb{R})$ gauge transformation with the transition functions $\{\bar{e}_a^\alpha\} \in GL\,(n,\mathbb{R})$ being the group elements. In other words, the connections in the two bases correspond to two points in two distinct sectors of a *single* gauge orbit in a *single* general linear group structure.

From eq. (9), the exterior derivative of $g_{\alpha\beta}$,

$$dg_{\alpha\beta} \;=\; d\left(e_\alpha^a e_\beta^b \bar{g}_{ab}\right) \;=\; -\,\omega^\gamma{}_\alpha e_\gamma^a e_\beta^b \bar{g}_{ab} - \omega^\gamma{}_\beta e_\alpha^a e_\gamma^b \bar{g}_{ab} \tag{11}$$

$$+ e^c_\alpha e^b_\beta \bar{\omega}^a{}_c \bar{g}_{ab} + e^a_\alpha e^c_\beta \bar{\omega}^b{}_c \bar{g}_{ab} + e^a_\alpha e^b_\beta d\bar{g}_{ab},$$

gives rise to the (obvious) relation:

$$(D_\omega g)_{\alpha\beta} \;=\; e^a_\alpha e^b_\beta \left(D_{\bar{\omega}} \bar{g}\right)_{ab}. \tag{12}$$

However, definition (4) implies that the metric (coordinate) tensor is a covariant-free object,

$$\left.\begin{aligned} dg_{\alpha\beta} &= -\omega^\gamma{}_\alpha g_{\gamma\beta} - \omega^\gamma{}_\beta g_{\gamma\alpha} \\ d\bar{g}_{ab} &= -\bar{\omega}^c{}_a \bar{g}_{cb} - \bar{\omega}^c{}_b \bar{g}_{ca} \end{aligned}\right\} \;\Rightarrow\; (D_\omega g)_{\alpha\beta} \;=\; \left(D_{\bar{\omega}} \bar{g}\right)_{ab} \;=\; 0. \tag{13}$$

It is precisely by this meaning that the connection was said to be *compatible* with the metric. Transforming to any other an-holonomic basis won't change anything because this would only be a mere gauge transformation. This is the property of *metricity*; the covariant exterior derivative of the metric tensor vanishes in *any* basis.

If $\{\boldsymbol{e}_a\}$ form an x-independent rectilinear system everywhere, namely $\bar{g} = \mathrm{diag}\,(p,q) =: \eta$ $(p+q = n)$, then, by the lower left equation in (13), $\bar{\omega}_{ab} = -\bar{\omega}_{ba}$, and the restriction to orthonormal frames is established as a symmetry structure with the (pseudo) rotational group (the isometry group of η) being the gauge group. In this case, for any element $o \in O\,(p,q)$,

$$\boldsymbol{e} \;\mapsto\; o\boldsymbol{e}, \qquad \eta \;\mapsto\; (o \otimes o)\,\eta = \eta, \quad \text{and} \quad \bar{\omega} \;\mapsto\; o\bar{\omega}o^{-1} + odo^{-1}. \tag{14}$$

The connection coefficients in their holonomic version are obtained in a standard manner by permuting the indices in $\partial_\alpha g_{\beta\gamma}$ and summing over with alternating signs,

$$\partial_\alpha g_{\beta\gamma} - \partial_\beta g_{\gamma\alpha} + \partial_\gamma g_{\alpha\beta} = -\omega_{[\alpha|\beta|\gamma]} + \omega_{[\beta|\gamma|\alpha]} - \omega_{[\gamma|\alpha|\beta]} - 2\omega_{\gamma\beta\alpha}. \tag{15}$$

Using Schouten's convention [3], $\{\alpha\beta\gamma\} = \alpha\beta\gamma - \beta\gamma\alpha + \gamma\alpha\beta$, eq. (15) can be put also in the compact form,

$$\omega_{\gamma\beta\alpha} = -\frac{1}{2}\left[\partial_{\{\alpha}\, g_{\,\beta\gamma\}} + \omega_{\{[\alpha|\beta|\gamma]\}}\right]. \tag{16}$$

In a holonomic basis, the anti-symmetric pieces in $\omega_{\{[\alpha|\beta|\gamma]\}}$ vanish, whence $\omega_{\gamma\beta\alpha}$ reduces to (minus) the Christoffel symbol with the upper index lowered. In its non-holonomic version formula (15) employs $\bar{g}$ and $\bar{\omega}$, instead of g and ω. In particular, in a rectilinear system the $\partial\bar{g}$'s vanish, in which case $\bar{\omega}$ is purely determined by the an-holonomy coefficients (see eqs. (39)-(40), section 2.4).

2.2 The emergence of non-metricity

Having seen that metricity is ingrained in definition (3), we realize that in order to avoid it we must modify the definition so as to make it less restrictive. The simplest sensible modification is to add a non-homogeneous term in the r.h.s. of the original definition:

$$\lim_{\Delta x \to 0} \left[\boldsymbol{e}_\alpha (x + \Delta x)|_x - \boldsymbol{e}_\alpha (x)|_x \right] = -\varpi^\beta{}_\alpha (x)\, \boldsymbol{e}_\beta (x)|_x - \boldsymbol{q}_\alpha (x)|_x \,, \tag{17}$$

where $\boldsymbol{q}_\alpha (x)$ is a vector-valued 1-form (note: $\varpi \neq \omega$). In our abbreviated notation,

$$d\boldsymbol{e}_\alpha = -\varpi^\beta{}_\alpha \boldsymbol{e}_\beta - \boldsymbol{q}_\alpha, \tag{18}$$

which means that the expansion of $d\boldsymbol{e}_\alpha$ in the basis $\{\boldsymbol{e}_\alpha\}$ should be *locally* corrected. As a consequence of this correction, the frames are no longer covariant-free (with respect to ϖ, of course), but instead we have:

$$(D_\varpi \boldsymbol{e})_\alpha = -\boldsymbol{q}_\alpha. \tag{19}$$

The $\boldsymbol{q}$-terms endow the underlying manifold with additional structure (more properly, with a sub-structure) which is obviously reflected in the form of the connection coefficients, see eq. (26) ahead. Since each shift $\boldsymbol{q}_\alpha$ in definition (17) is a *vector-valued* object, it can be expanded in the basis $\{\boldsymbol{e}_\alpha\}$:

$$\boldsymbol{q}_\alpha = Q^\beta{}_\alpha \boldsymbol{e}_\beta \quad \Rightarrow \quad Q_{\alpha\beta} = \boldsymbol{e}_\alpha \cdot \boldsymbol{q}_\beta, \tag{20}$$

where $Q^{\beta}{}_{\alpha}$ is a coordinate tensor 1-form. From eqs. (18) and (20), definition (3) is back recovered with respect to the composed connection $\omega^{\beta}{}_{\alpha} = \varpi^{\beta}{}_{\alpha} + Q^{\beta}{}_{\alpha}$.

Eq. (18) gives rise to non-metricity through

$$dg_{\alpha\beta} = -\varpi_{\alpha\beta} - \varpi_{\beta\alpha} - \boldsymbol{q}_{\alpha} \cdot \boldsymbol{e}_{\beta} - \boldsymbol{e}_{\alpha} \cdot \boldsymbol{q}_{\beta} \;\Rightarrow\; (D_{\varpi} g)_{\alpha\beta} = -Q_{(\alpha\beta)} \tag{21}$$

with $Q_{(\alpha\beta)} := \boldsymbol{q}_{\alpha} \cdot \boldsymbol{e}_{\beta} + \boldsymbol{e}_{\alpha} \cdot \boldsymbol{q}_{\beta}$. Definition (17) must hold also in any non-holonomic basis,

$$d\boldsymbol{e}_{a} = -\bar{\varpi}^{b}{}_{a} \boldsymbol{e}_{b} - \boldsymbol{q}_{a} \tag{22}$$

and the non-metricity in its an-holonomic version reads:

$$d\bar{g}_{ab} = -\bar{\varpi}_{ab} - \bar{\varpi}_{ba} - \boldsymbol{q}_{a} \cdot \boldsymbol{e}_{b} - \boldsymbol{e}_{a} \cdot \boldsymbol{q}_{b} \;\Rightarrow\; (D_{\bar{\varpi}} \bar{g})_{ab} = -Q_{(ab)} \tag{23}$$

with $Q_{(ab)} := \boldsymbol{q}_{a} \cdot \boldsymbol{e}_{b} + \boldsymbol{e}_{a} \cdot \boldsymbol{q}_{b}$. Consistency then requires that the $\boldsymbol{q}$'s in the holonomic and the an-holonomic bases should be interrelated via $\boldsymbol{q}_{\alpha} = e^{a}_{\alpha} \boldsymbol{q}_{a}$. This requirement, in turn, implies that the n-beins are not necessarily covariant free because

$$(D_{\varpi\bar{\varpi}} e)^{a}_{\alpha} = -\boldsymbol{q}_{\alpha} \cdot \boldsymbol{e}^{a} + \boldsymbol{e}_{\alpha} \cdot \boldsymbol{q}^{a} = g^{ab} e^{c}_{\alpha} Q_{[bc]}, \tag{24}$$

and $Q_{[ab]}$ may not vanish. Nevertheless, the four extra terms generated in $dg_{\alpha\beta}$ due to (24) cancel each other exactly, therefore eqs. (11)-(12) maintain their original form, this time, however, with non-vanishing non-metricity:

$$(D_{\varpi} g)_{\alpha\beta} = e^{a}_{\alpha} e^{b}_{\beta} (D_{\bar{\varpi}} \bar{g})_{ab} = -e^{a}_{\alpha} e^{b}_{\beta} Q_{(ab)} \equiv -Q_{(\alpha\beta)}. \tag{25}$$

A spacetime endowed with connection and metric structure that satisfy eq. (25) is called *post-Riemannian*, and is frequently denoted by (L_n, g). Following the method used in getting eq. (15), the connection coefficients in their holonomic version now satisfy:

$$\begin{aligned} &\partial_{\alpha} g_{\beta\gamma} - \partial_{\beta} g_{\gamma\alpha} + \partial_{\gamma} g_{\alpha\beta} + Q_{\alpha(\beta\gamma)} - Q_{\beta(\gamma\alpha)} + Q_{\gamma(\alpha\beta)} \\ &\qquad = -\varpi_{[\alpha|\beta|\gamma]} + \varpi_{[\beta|\gamma|\alpha]} - \varpi_{[\gamma|\alpha|\beta]} - 2\varpi_{\gamma\beta\alpha}. \end{aligned} \tag{26}$$

$$\Rightarrow \quad \varpi_{\gamma\beta\alpha} \;=\; -\frac{1}{2}\left[\partial_{\{\alpha}\, g_{\beta\gamma\}} + Q_{\{\alpha(\beta\gamma)\}} + \varpi_{\{[\alpha|\beta|\gamma]\}}\right]. \tag{27}$$

This is the well-known Schouten formula, see [3, page 132], or [4, eq. (3.10.8)].[4] Here, and contrary to the case of $\omega_{\gamma\beta\alpha}$ - see eq. (16) - all the n^3 degrees of freedom in the connection are employed, reflecting the removal of the metricity constraint.

It is important to notice that, due to (18), the covariant exterior derivative of a vector field in an (L_n, g) is no longer the coordinate vector of its absolute differential,

$$d\boldsymbol{v} \;=\; (D_\varpi v)^\alpha\, \boldsymbol{e}_\alpha - v^\alpha \boldsymbol{q}_\alpha \tag{28}$$

(compare this with eq. (7)); yet, $d\boldsymbol{v}$ is still an invariant because $\boldsymbol{q}_\alpha$ is covariant. In the more general case, the absolute differential of a tensor p-form (recall eq. (8)) involves additional contractions,

$$\begin{aligned} d\boldsymbol{t}_p \;&=\; (D_\varpi t_p)^{\alpha_1\alpha_2\cdots}{}_{\beta_1\beta_2\cdots}\boldsymbol{e}_{\alpha_1\alpha_2\cdots}{}^{\beta_1\beta_2\cdots} + (-1)^{p+1}\, t_p{}^{\gamma\alpha_2\cdots}{}_{\beta_1\beta_2\cdots} \wedge Q^{\alpha_1}{}_{\gamma}\boldsymbol{e}_{\alpha_1\alpha_2\cdots}{}^{\beta_1\beta_2\cdots} + \cdots \\ &+\; (-1)^p\, t_p{}^{\alpha_1\alpha_2\cdots}{}_{\gamma\beta_2\cdots} \wedge Q^{\gamma}{}_{\beta_1}\boldsymbol{e}_{\alpha_1\alpha_2\cdots}{}^{\beta_1\beta_2\cdots} + \cdots . \end{aligned} \tag{29}$$

In particular, substituting $\boldsymbol{g} = g_{\alpha\beta}\boldsymbol{e}^{(\alpha\beta)}$ for $\boldsymbol{t}_0$ in (29) ($\boldsymbol{e}^{(\alpha\beta)}$ is a symmetric basis for $T\Omega^{\otimes 2}$, the bundle of rank-2 tensors over Ω) yields,

$$d\boldsymbol{g} \;=\; (D_\varpi g)_{\alpha\beta}\, \boldsymbol{e}^{(\alpha\beta)} + g_{\gamma\beta}Q^{\gamma}{}_{\alpha}\boldsymbol{e}^{(\alpha\beta)} + g_{\alpha\gamma}Q^{\gamma}{}_{\beta}\boldsymbol{e}^{(\alpha\beta)} \;=\; 0. \tag{30}$$

Hence, despite the fact that the *metric tensor* is not covariant-free, its absolute differential vanishes just like in the Riemannian case. In view of this result, and in view of the relation between the absolute differential of an object, and its covariant exterior derivative in general, we shall next re-examine the concept of parallel displacement in an (L_n, g).

[4] In general, $\varpi_{[\alpha|\beta|\gamma]} \neq 0$ even in a holonomic basis due to the presence of torsion, see eq. (45).

2.3 A digression on parallel displacements in an (L_n, g)

In the following we shall confine ourselves to holonomic bases, but our arguments, of course, apply to an-holonomic bases as well. Let $\boldsymbol{v} = v^\alpha \boldsymbol{e}_\alpha$ be a vector field whose norm is given by $v^2 = \boldsymbol{v} \cdot \boldsymbol{v} = v^\alpha v^\beta g_{\alpha\beta}$. The exterior derivative of v^2 reads:

$$d\left(\boldsymbol{v} \cdot \boldsymbol{v}\right) = \left(D_\varpi v\right)^\alpha v^\beta g_{\alpha\beta} + v^\alpha \left(D_\varpi v\right)^\beta g_{\alpha\beta} - v^\alpha v^\beta Q_{(\alpha\beta)} \tag{31}$$

$$= D_\varpi \left(v^\alpha v^\beta g_{\alpha\beta}\right) = D_\varpi \left(\boldsymbol{v} \cdot \boldsymbol{v}\right); \tag{32}$$

it displays in a naive manner the breakdown of the Leibnitz rule with respect to *covariant* exterior differentiation, because

$$D_\varpi \left(\boldsymbol{v} \cdot \boldsymbol{v}\right) \neq \left(D_\varpi \boldsymbol{v}\right) \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \left(D_\varpi \boldsymbol{v}\right). \tag{33}$$

From eq. (28), the inequality (33) stands in contrast with $d\left(\boldsymbol{v} \cdot \boldsymbol{v}\right) = d\boldsymbol{v} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot d\boldsymbol{v}$.

The parallel displacement of a vector $\boldsymbol{v}$ along a curve with tangent vector $\boldsymbol{u}$ is usually identified with the evaluation of $D_\varpi \boldsymbol{v}$ on $\boldsymbol{u}$; $\boldsymbol{v}_P$ is therefore said to be parallel if it satisfies $D_\varpi \boldsymbol{v}_P = 0$. The inequality (33), however, raises the question whether parallel displacement is a reliable concept in an (L_n, g). For, suppose that $\boldsymbol{v}_P$ has a null norm, namely $\boldsymbol{v}_P \cdot \boldsymbol{v}_P = v_P^\alpha v_P^\beta g_{\alpha\beta} = 0$. Then, by eqs. (31)-(32), and due to $\left(D_\varpi v_P\right)^\alpha = 0$, we have:

$$0 = D_\varpi \left(\boldsymbol{v}_P \cdot \boldsymbol{v}_P\right) = -v_P^\alpha v_P^\beta Q_{(\alpha\beta)} \tag{34}$$

but, apart from one special case (the Weyl-Cartan spacetime), $Q_{(\alpha\beta)} \neq f \times g_{\alpha\beta}$, where f is a 1-form, leading to an apparent inconsistency.

Therefore, instead of identifying a parallel field $\boldsymbol{v}_P$ with a field whose covariant exterior derivative vanishes, it seems more appropriate here to identify it with a field whose absolute differential vanishes, namely $d\boldsymbol{v}_P = 0$. $\boldsymbol{v}_P$ is then said to be *absolutely closed*. For example,

metric tensor $\boldsymbol{g}$ in eq. (30) is absolutely closed, but not covariantly closed. Of course, in the absence of non-metricity, in a metric spacetime, the two concepts of covariant closure and absolute closure become one and the same thing.

However, in the spirit of eq. (19) we have:

$$D_{\varpi}\boldsymbol{v} := D_{\varpi}\left(v^{\alpha}\boldsymbol{e}_{\alpha}\right) = \left(D_{\varpi}v\right)^{\alpha}\boldsymbol{e}_{\alpha} + v^{\alpha}\left(D_{\varpi}\boldsymbol{e}\right)_{\alpha} = \left(D_{\varpi}v\right)^{\alpha}\boldsymbol{e}_{\alpha} - v^{\alpha}\boldsymbol{q}_{\alpha} \equiv d\boldsymbol{v}. \tag{35}$$

It is therefore only in the sense of the *definition* $D_{\varpi}\boldsymbol{v} := \left(D_{\varpi}v\right)^{\alpha}\boldsymbol{e}_{\alpha}$ that the inequality (33) holds true, and that the concepts of absolute closure and covariant closure are different from each other. From the point of view of eq. (35) there is no such difference because we merely decomposed a Riemannian connection into a post-Riemannian one plus a compensation term; in this respect a parallel field is unambiguously defined through $D_{\varpi}\boldsymbol{v}_P = D_{\omega}\boldsymbol{v}_P = d\boldsymbol{v}_P = 0$.

2.4 The inclusion of torsion, and the structural identities

From this point, and throughout the rest of this work, we will no longer distinguish between different types of bases. We shall instead work in a genuine *non-holonomic* basis (the one attached to a local observer), and we shall employ *Greek* indices for that basis; we will not mark the connection with a bar, and we shall use the (bare) letter g for the local metric.

It is common to think of a coframe element ϑ^{α} simply as an object 'dual' to the frame element $\boldsymbol{e}_{\alpha}$. This interpretation, however, might mislead: coframes, as opposed to frames, are by definition Grassmann elements and are therefore automatically annihilated by d^2. In order to sharpen this point, we return for a moment to Riemannian geometry.

A second application of d at the frames generates the curvature 2-form,

$$dd\boldsymbol{e}_{\alpha} \;=\; -\left(d\omega^{\beta}{}_{\alpha} + \omega^{\gamma}{}_{\alpha}\wedge\omega^{\beta}{}_{\gamma}\right)\boldsymbol{e}_{\beta} \;=\; -R^{\beta}{}_{\alpha}\left(\omega\right)\boldsymbol{e}_{\beta}, \tag{36}$$

and in general, the $(2k)$-th application gives:

$$d^{2k}\boldsymbol{e}_{\alpha} \;=\; (-1)^{k}\, R^{\beta}{}_{\alpha}\wedge R^{\gamma}{}_{\beta}\wedge\cdots\wedge R^{\delta}{}_{\gamma}\boldsymbol{e}_{\delta}, \tag{37}$$

where $R \;\equiv\; R\,(\omega)$.[5] Therefore, $\left(d^{2k}\boldsymbol{e}_{\alpha}\right)\otimes\boldsymbol{e}^{\alpha}$ is the k-th order polynomial in the rotational curvature (given up to a sign), and $\left(d^{2k}\boldsymbol{e}_{\alpha}\right)\cdot\boldsymbol{e}^{\alpha}$ is simply its trace. Similarly, if $\boldsymbol{v}_p$ is a vector p-form, then, by formula (1), and from (36): $d^2\boldsymbol{v}_p = -v_p^{\alpha}\wedge R^{\beta}{}_{\alpha}\,(\omega)\,\boldsymbol{e}_{\beta}$.

Since a coframe system is required to play a role *similar* to that played by a frame system, the exterior derivative of a coframe element ϑ^{α} has to be given by

$$d\vartheta^{\alpha} = -\vartheta^{\beta}\wedge\omega^{\alpha}{}_{\beta}. \tag{38}$$

On the other hand, being a Grassmannian element, and because d is a raising operator in a de-Rham complex, ϑ^{α} necessarily satisfies the relation

$$d\vartheta^{\alpha} =: \frac{1}{2}C^{\alpha}{}_{[\beta\gamma]}\vartheta^{\beta}\wedge\vartheta^{\gamma}, \tag{39}$$

where $d\vartheta^{\alpha} =: C^{\alpha}$ is the *an-holonomy* 2-form.[6] Eqs. (38) and (39) imply the equivalence

$$\omega_{[\beta|\alpha|\gamma]} \equiv C_{\alpha[\beta\gamma]}. \tag{40}$$

Now, in Contrast with $d^2\boldsymbol{e}\neq 0$, we have, by definition, $d^2\vartheta = 0$. Hence, when d^2 is applied to coframes, it generates *constraints*, rather than new objects as in eqs. (36)-(37). For example: if d is applied at eq. (38) it yields the constraint

$$0 \;=\; dd\vartheta^{\alpha} \;=\; \vartheta^{\beta}\wedge R^{\alpha}{}_{\beta}\,(\omega)\,; \tag{41}$$

one recognizes formula (41) as the 1-st Bianchi identity in a Riemannian spacetime V_n. On the other hand, if d is applied at eq. (39), it gives

$$dC^{\alpha}{}_{[\beta\gamma]} = \frac{1}{2}\left(C^{\alpha}{}_{[\beta\delta]}C^{\delta}{}_{[\gamma\epsilon]} - C^{\alpha}{}_{[\gamma\delta]}C^{\delta}{}_{[\beta\epsilon]}\right)\vartheta^{\epsilon}, \tag{42}$$

[5]Note that $R\,(\omega)\neq D_{\omega}\omega\,(=d\omega+\omega\wedge\omega+\omega\wedge\omega)$; instead, $R\,(\omega)=d\omega+\omega\wedge\omega=d\omega+\frac{1}{2}\,[\![\omega,\omega]\!]$.

[6]C^{α} is not a covariant object; if ϑ transforms as $\vartheta\mapsto\vartheta\ell^{-1}$, then C transforms as $C\mapsto C\ell^{-1}-\vartheta\wedge d\ell^{-1}$.

which is a variant of the Maurer-Cartan equation associated with the bundle of orthonormal frames. However, had we applied d at eq. (39), and used eq. (38) (instead of eq. (39)) for $d\vartheta$, we would have obtain:

$$(D_{\omega} C^{\alpha})_{[\beta\gamma]} = 0, \tag{43}$$

which is, in fact, that particular condition on the an-holonomy, playing a role parallel to the metricity condition on the metric. This is the *holonomicity condition*. Notice that the covariant differentiation of C^{α} is taken with respect to its *basespace* indices.

We now return to the post-Riemannian case. Owing to the similar role played by frames and coframes, a consistent extension of the Riemannian relation (38) would be to add an x-dependent shift in its r.h.s.:

$$d\vartheta^{\alpha} = -\vartheta^{\beta} \wedge \varpi^{\alpha}{}_{\beta} - T^{\alpha}. \tag{44}$$

Should definition (44) hold in any given basis, the *torsion* - $\boldsymbol{T} := T^{\alpha} \boldsymbol{e}_{\alpha}$ - must behave as a vector 2-form. This is the place to comment that in a non-holonomic basis, the distinction between the base sector and the fiber sector becomes rather vague. For example, the index α in $d\vartheta^{\alpha}$ in (44) is essentially a fiberspace index although the same index in ϑ^{α} is a basespace index; see also eq. (43), and eq. (46) below, where a gauge-sector differentiation is applied to basespace indices.

Combining eqs. (39) and (44), the torsion and the an-holonomy give rise to the following decomposition of the antisymmetric piece in the connection:

$$\varpi_{[\beta|\alpha|\gamma]} = C_{\alpha[\beta\gamma]} + T_{\alpha[\beta\gamma]}. \tag{45}$$

Furthermore, from eqs. (39) and (44),

$$(D_{\varpi} C^{\alpha})_{[\beta\gamma]} \wedge \vartheta^{\beta} \wedge \vartheta^{\gamma} = C^{\alpha}{}_{[\beta\gamma]} T^{[\beta} \wedge \vartheta^{\gamma]}, \tag{46}$$

which signifies the removal of the constraint on the an-holonomy in a manner similar to the removal of the metricity constraint on the metric. This is *non-holonomicity*, the an-holonomy counterpart of non-metricity.

Let $h = h_\alpha \vartheta^\alpha$ be a 1-form. Making use of definition (44), the detailed exterior derivative of h in a spacetime with torsion is given by

$$dh = (D_\varpi h)_\alpha \wedge \vartheta^\alpha - h_\alpha T^\alpha, \tag{47}$$

and $d^2 h = 0$ whatsoever. By construction, $(D_\varpi h)_\alpha$ treats h_α in the same manner it treats it in $\boldsymbol{h} = h_\alpha \boldsymbol{e}^\alpha$. Generally speaking, in the presence of torsion, the absolute differential of a scalar-valued p-form $t_p = t_{\alpha_1 \cdots \alpha_p} \vartheta^{\alpha_1} \wedge \cdots \wedge \vartheta^{\alpha_p}$, taking into account also the derivative of the Grassmann basis, gets an additional torsional contribution:

$$dt_p \;=\; (D_\varpi t_p)_{\alpha_1 \cdots \alpha_p} \vartheta^{\alpha_1} \wedge \cdots \wedge \vartheta^{\alpha_p} - p\, t_{\alpha_1 \alpha_2 \cdots \alpha_p} T^{\alpha_1} \wedge \vartheta^{\alpha_2} \wedge \cdots \wedge \vartheta^{\alpha_p}; \tag{48}$$

here $(D_\varpi t_p)_{\alpha_1 \cdots \alpha_p} \vartheta^{\alpha_1} \wedge \cdots \wedge \vartheta^{\alpha_p}$ treats the (basespace) components of t_p as if they were the components of an anti-symmetric tensor field $\boldsymbol{t}_p = t_{\alpha_1 \cdots \alpha_p} \boldsymbol{e}^{[\alpha_1 \cdots \alpha_p]}$.

As we have already argued, in contradistinction with forms, d^2 fails to vanish on frames. Instead, in the presence of non-vanishing non-metricity,

$$dd\boldsymbol{e}_\alpha = -R^\beta{}_\alpha\,(\varpi)\,\boldsymbol{e}_\beta - (D_\varpi \boldsymbol{q})_\alpha \tag{49}$$

(compare this with eq. (36)). From eq. (36), the (tensorial) equality $-D_\varpi \boldsymbol{q} = \boldsymbol{R}\,(\varpi)$ corresponds to $\boldsymbol{R}\,(\omega) = 0$.

Consider the vector 1-form $\boldsymbol{\pi} = \vartheta^\alpha \boldsymbol{e}_\alpha$. Its square in a coordinate basis is just the usual line element, $\pi^2 = \boldsymbol{\pi} \cdot \boldsymbol{\pi} = dx^\alpha dx^\beta g_{\alpha\beta}$.[7] In a Riemannian spacetime $d\boldsymbol{\pi}$ identically vanishes,

[7]Note that the line-element, like any cone, is a transvection field [3], not a tensorial quantity.

and therefore $d\boldsymbol{\pi}^2$ absolutely closes; in this sense ϑ^α and $\boldsymbol{e}_\alpha$ are bases 'dual' to one another. However, in an (L_n, g), in the presence of torsion and non-metricity,

$$d\boldsymbol{\pi} = -T^\alpha \boldsymbol{e}_\alpha + \vartheta^\alpha \wedge \boldsymbol{q}_\alpha, \tag{50}$$

whence we may conclude that $d\boldsymbol{\pi} \neq 0$. What seems to be an exceptional case is when the torsion $\boldsymbol{T}$ and the shift $\boldsymbol{q}_\alpha$ are mutually interrelated through

$$\boldsymbol{T} = \vartheta^\alpha \wedge \boldsymbol{q}_\alpha. \tag{51}$$

In fact, condition (51) is more than an exceptional case. As we shall explain in section 3.1 below, it is a *consistency requirement* in a theory in which a post-Riemannian spacetime is realized as a deformed Riemannian spacetime. Due to eq. (51), frames and coframes remain dual concepts, and the line element π^2 is kept absolutely closed, also in an (L_n, g).

There are three basic structural identities in an (L_n, g) - the so-called *Bianchi identities*. The second of which, and probably the most familiar one, reads:

$$dR^\beta{}_\alpha(\varpi) = -R^\beta{}_\gamma(\varpi) \wedge \varpi^\gamma{}_\alpha + R^\gamma{}_\alpha(\varpi) \wedge \varpi^\beta{}_\gamma \quad \Rightarrow \quad (D_\varpi R(\varpi))^\beta{}_\alpha = 0. \tag{52}$$

In a Riemannian spacetime, the same identity for $R(\omega)$ is equivalent to the absolute closure of the curvature tensor, $d\boldsymbol{R} = 0$. In a post-Riemannian spacetime, however, the curvature no longer absolutely closes; instead we find the tensorial equation

$$d\boldsymbol{R} = [\![\boldsymbol{R}, \boldsymbol{Q}]\!], \tag{53}$$

as one may directly infer from formula (29). The other two Bianchi identities, the 1-st and the 3-rd, emerge from the nilpotency of d on ϑ^α and $g_{\alpha\beta}$, respectively:

$$0 = dd\vartheta^\alpha = \vartheta^\beta \wedge R^\alpha{}_\beta - (D_\varpi T)^\alpha, \tag{54}$$

$$0 = ddg_{\alpha\beta} = (D_\varpi Q)_{(\alpha\beta)} + R_{(\alpha\beta)}. \tag{55}$$

In particular, from identity (54), the transvected curvature $R_{\alpha\beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta$ is purely of a post-Riemannian origin, and from identity (55), so does the curvature-trace, $g^{\alpha\beta} R_{\alpha\beta}$.[8]

Finally, being a vector 1-form, $\boldsymbol{\pi}$ fails to be annihilated by d^2. Instead we find:

$$\begin{aligned} dd\boldsymbol{\pi} &= \vartheta^\alpha \wedge dd\boldsymbol{e}_\alpha = -\vartheta^\beta \wedge R^\alpha{}_\beta(\varpi)\, \boldsymbol{e}_\alpha - \vartheta^\alpha \wedge (D_\varpi \boldsymbol{q})_\alpha \\ &= -(D_\varpi T)^\alpha\, \boldsymbol{e}_\alpha - \vartheta^\alpha \wedge (D_\varpi \boldsymbol{q})_\alpha\,, \end{aligned} \tag{56}$$

where we exploited eqs. (1), (49), and identity (54). As in eq. (49), the (vectorial 3-form) equality $-D_\varpi \boldsymbol{T} = \vartheta \wedge D_\varpi \boldsymbol{q}$ corresponds to $\vartheta \wedge \boldsymbol{R}(\omega) = 0$.

3 The post-Riemannian merger of internal symmetries with spacetime symmetries

3.1 The deformation criterion and the basespace-framespace splitting

The invariant object $\boldsymbol{\pi} = \vartheta^\alpha \boldsymbol{e}_\alpha$ whose square in a coordinate basis is the infinitesimal line element, is a vector 1-form. The "tensorial power" of this object would therefore be a p-form:

$$\boldsymbol{\pi}_p = \vartheta^{\alpha_1} \wedge \cdots \wedge \vartheta^{\alpha_p} \boldsymbol{e}_{\alpha_1} \otimes \cdots \otimes \boldsymbol{e}_{\alpha_p}. \tag{57}$$

Now, let us recall that in order to form the square of $\boldsymbol{\pi}$ we employed a dot product that eventually ended up in a *transvection*. But this wasn't an 'ordinary' dot product because two 1-forms were mapped to a scalar (the infinitesimal line element). Let us then formulate this mapping in a precise language. There exists a map, say *odot* $\odot$,

$$\odot : \ \wedge^p \left(T\Omega^{\otimes p}\right) \times \wedge^p \left(T\Omega^{\otimes p}\right) \ \mapsto \ \wedge^0(\Omega)\,, \tag{58}$$

[8]See in this respect the Appendix B.4 in [4], where the irreducible pieces of the curvature in an (L_n, g) are classified; see also the related discussions in [5, 6].

that takes two rank-p tensor p-forms (invariants) into a transvection (an invariant), and whose realization on a pair of $\boldsymbol{\pi}_p$'s is given by

$$\boldsymbol{\pi}_p \odot \boldsymbol{\pi}_p \stackrel{\text{def}}{=} \vartheta^{\alpha_1}\vartheta^{\beta_1}g_{\alpha_1\beta_1}\cdots\vartheta^{\alpha_p}\vartheta^{\beta_p}g_{\alpha_p\beta_p} = \pi_p^2\,, \quad \text{and so} \quad \pi_p^2 = \left(\pi^2\right)^p . \tag{59}$$

In a coordinate basis, π_p^2 makes up a "line-element" on $T\Omega^{\otimes p}$.

In a Riemannian spacetime metricity guarantees that the line-element remains invariant against parallel displacement. The post-Riemannian analogue would therefore be the absolute closure of $\boldsymbol{\pi}_p$ (any $p \geq 1$), and this is identically fulfilled only if

$$\boldsymbol{T} = \vartheta^\alpha \wedge \boldsymbol{q}_\alpha, \quad \text{or} \quad T^\alpha = \vartheta^\beta \wedge Q^\alpha{}_\beta, \tag{60}$$

which is precisely condition (51). Eq. (60) establishes the symmetry structure induced on the frames, with the gauge group $L\left(n, \mathbb{R}\right)$ of linear transformations[9], on a connection-deformation basis: it is *the* necessary requirement that the deformation of the connection, as was presented in eq. (44), where the torsion was introduced, will be compatible with the deformation of the connection as was presented in eq. (18), where non-metricity was introduced:

$$\begin{aligned} d\boldsymbol{e}_\alpha &= -\underbrace{\left(\varpi^\beta{}_\alpha + Q^\beta{}_\alpha\right)}_{=\,\omega^\beta{}_\alpha}\boldsymbol{e}_\beta \quad \Leftrightarrow \quad \boldsymbol{q}_\alpha \;=\; Q^\beta{}_\alpha\boldsymbol{e}_\beta\,, \\ d\vartheta^\alpha &= -\vartheta^\beta \wedge \underbrace{\left(\varpi^\alpha{}_\beta + Q^\alpha{}_\beta\right)}_{=\,\omega^\alpha{}_\beta} \quad \Leftrightarrow \quad T^\alpha \;=\; \vartheta^\beta \wedge Q^\alpha{}_\beta\,. \end{aligned} \tag{61}$$

Therefore, in a picture in which a post-Riemannian spacetime is regarded as a deformed Riemannian spacetime, eq. (60), the so-called *deformation criterion*, becomes a constitutive equation, saying that the torsion furnishes an anti-symmetric piece for the deformation term. On the other hand, since $Q_{(\alpha\beta)} := Q^\gamma{}_\alpha g_{\gamma\beta} + Q^\gamma{}_\beta g_{\gamma\alpha}$, the non-metricity furnishes a symmetric

[9]Here and in what follows the linear frame-transformation group $L\left(n, \mathbb{R}\right) \subseteq GL\left(n, \mathbb{R}\right)$ is assumed to be larger than $O\left(n\right)$.

piece. The coordinate tensor 1-form $\varphi^{\beta}{}_{\alpha} = -Q^{\beta}{}_{\alpha}$ is known by the name "*distortion 1-form*"; it will soon play a central role in a merging theory, as a host for internal symmetries.

The fact that we are dealing with a deformation process has additional implication that merits few words. Usually, the extension of the symmetry structure follows from an extension of the gauge group by adding new generators to the generating Lie algebra (this, of course, may also require change of representation). The connection 1-form of the extended structure takes values in the extended algebra, and those pieces with values in the *added* generators are responsible for the non-linear behavior of the (whole) connection under a transformation generated in the extension. This is, however, *not* an appropriate description when it comes to the *geometrical* formulation of the symmetry structure associated with the frame bundle. To see this explicitly, note that the non-linear terms generated in the transformation of $\varpi^{\beta}{}_{\alpha}$ under $L(n, \mathbb{R})$ are coming *solely* from the rotational piece, since $\boldsymbol{Q} = -D_{\varpi}\boldsymbol{g}$ is a true tensor, and since the derivatives (that generate the non-linear terms) are found only in the Christoffel symbol. Despite this solid fact, we will soon show that the *merger* of spacetime with internal degrees of freedom *does* impose non-linear behavior on the distortion tensor $\boldsymbol{Q}$, but it is done in a manner that has nothing to do with frame transformations.

The next step that we take, which is crucial to subsequent developments, is to factor out the basespace part in $\boldsymbol{q}_{\alpha}$,

$$\boldsymbol{q}_{\alpha} =: -A \otimes \mathfrak{q}_{\alpha} =: -A \otimes \mathcal{Q}^{\beta}{}_{\alpha}\boldsymbol{e}_{\beta} \tag{62}$$

with a 1-form A ($A = A_{\gamma}\vartheta^{\gamma}$), and a vector-valued 0-form $\mathfrak{q}_{\alpha}$.[10] We have deliberately invoked a tensor product in eq. (62), for we will soon realize that the 1-form A may possess internal degrees of freedom *without* essentially affecting the basic geometrical structure of spacetime.

[10]The splitting in (62) is natural in the sense that the frame bundle is *locally* a tensor product between the basespace and the frame fibers.

Hence, from now on, A is assumed to be a matrix-valued field, $A = A^i{}_j$. For convenience, however, we shall often suppress the indices this matrix (or any function of it) carries.

Employing the deformation criterion, eq. (60), and as a result of the splitting in (62), the torsion takes the form

$$\boldsymbol{T} = (A \wedge \vartheta^\alpha) \otimes \mathbf{q}_\alpha , \tag{63}$$

and the components of the non-metricity tensor read:

$$Q_{(\alpha\beta)} = -A \otimes \mathcal{Q}_{(\alpha\beta)} \quad \text{with} \quad \mathcal{Q}_{(\alpha\beta)} = \boldsymbol{e}_\alpha \cdot \mathbf{q}_\beta + \boldsymbol{e}_\beta \cdot \mathbf{q}_\alpha . \tag{64}$$

In what follows, we shall decompose the post-Riemannian connection as $\varpi^\beta{}_\alpha := \omega^\beta{}_\alpha + \varphi^\beta{}_\alpha$ (instead of decomposing the Riemannian connection as $\omega^\beta{}_\alpha = \varpi^\beta{}_\alpha + Q^\beta{}_\alpha$). From the left equations in (61), and due to (62), the distortion 1-form splits as:

$$\varphi^\beta{}_\alpha = -Q^\beta{}_\alpha = A \otimes \mathcal{Q}^\beta{}_\alpha . \tag{65}$$

Notice that, as opposed to $\omega^\beta{}_\alpha \wedge \omega^\alpha{}_\beta \equiv 0$, the product term $\varphi^\beta{}_\alpha \wedge \varphi^\alpha{}_\beta$ vanishes *only for an Abelian* A. Therefore, from now on, since none of the geometrical objects that depend on A (gradely) commute with each other, much care is required when handling calculations; in particular, one must keep an eye on the *order* in which terms are arranged in a product. For example: because our convention has been to put tensor bases always to the right, the connection in a covariant exterior derivative should always appear to the *right* of the object being covariantly differentiated.

3.2 Embedding a $\mathbb{C}$-valued non-Abelian internal symmetry structure within the non-metric sector of an (L_n, g)

As we have already implied in section 3.1, we intend to endow A with internal degrees of freedom. More precisely, we wish to associate $\varphi(A)$ with a connection 1-form in a certain internal space, equipped with a local symmetry structure acted upon by a complex-valued gauge group $U(N, \mathbb{C})$, where N being the dimensionality of U.[11] Hence we require that under the action of any element u in that group,[12]

$$\varphi(A) \quad \mapsto \quad u\varphi(A)\, u^{-1} + udu^{-1} \otimes \mathbf{1}_n, \tag{66}$$

where $\mathbf{1}_n$ is the identity element in frame space (recall the $\boldsymbol{\varphi}(A)$ is a frame-space tensor). This requirement originates from the mapping

$$\boldsymbol{q}(A) \quad \mapsto \quad u\boldsymbol{q}(A)\, u^{-1} - udu^{-1} \otimes \boldsymbol{e}, \tag{67}$$

see section 3.3 for the detailed analysis.

Due to this association, and as a result of the mapping rule postulated in (66), the "encrusted" post-Riemannian connection[13]

$$\varpi(\omega, \varphi) = \omega \otimes \mathbf{1}_{\rho(U)} + \varphi(A), \tag{68}$$

behaves as an 'hyper' connection, gauging independently two local symmetry structures in a spliced fiber bundle (here one symmetry refers to the frames, the other refers to the internal space where A lives):

$$\begin{aligned} \forall\, \ell \in L: \quad \varpi \quad &\mapsto \quad \ell\varpi\ell^{-1} + \ell d\ell^{-1} \otimes \mathbf{1}_{\rho(U)} \\ \forall\, u \in U: \quad \varpi \quad &\mapsto \quad u\varpi u^{-1} + udu^{-1} \otimes \mathbf{1}_n. \end{aligned} \tag{69}$$

[11] Content with little, we could have taken U to be $\mathbb{R}$-valued, but we can do better than that.

[12] Focusing here only on the general idea, we shall suppress all kinds of indices in the meantime.

[13] $\rho(U)$ means that U is in the representation ρ; $\mathbf{1}_{\rho(U)}$ is the unit element in that representation.

This type of connection establishes a connectivity on the so-called *foliar bundle*, a product bundle whose symmetry structures interlace in a non-trivial manner. The detailed paradigm of this merger setup, in its most general form, is given in [7].

Let $O(p,q)$ be generated by $\mathfrak{o}(p,q)$ $(p+q=n)$, let $L(n,\mathbb{R})$ be generated by $\mathfrak{l}(n)$, and let $U(N,\mathbb{C})$ be generated by $\mathfrak{u}(N)$. The algebraic structure of ϖ, as suggested by formulas (66)-(69), can be put in the pictorial form:

$$\underbrace{\mathfrak{o}(p,q) \oplus \overbrace{[\mathfrak{l}(n)/\mathfrak{o}(p,q)]}^{\otimes\,\mathfrak{u}(N)}}_{\mathfrak{l}(n)} \tag{70}$$

meaning that ϖ takes its values in the direct sum,

$$\varpi \in \left[\mathfrak{o}(p,q) \otimes \mathbf{1}_{\rho(U)}\right] \oplus \left[\frac{\mathfrak{l}(n)}{\mathfrak{o}(p,q)} \otimes \mathfrak{u}(N)\right]. \tag{71}$$

In decomposition (71), and in the descriptive scheme (70), $\mathfrak{o}(p,q)$ generates pseudo-rotations, and the quotient space $\mathfrak{s}(n) = \mathfrak{l}(n)/\mathfrak{o}(p,q)$ generates shears and dilations. A detailed analysis of the algebra $\mathfrak{gl}(n) \supseteq \mathfrak{l}(n)$ is given in [9]; here we only mention that the different sectors in this algebra satisfy the inclusion relations

$$[\mathfrak{o},\mathfrak{o}] \subset \mathfrak{o}, \qquad [\mathfrak{s},\mathfrak{s}] \subset \mathfrak{o}, \quad \text{and} \quad [\mathfrak{o},\mathfrak{s}] \subset \mathfrak{s}. \tag{72}$$

Substituting $\varpi = \omega + \varphi$ in $\mathcal{R}(\varpi) = d\varpi + \varpi \wedge \varpi$ gives:

$$\mathcal{R}(\varpi) = R(\omega) + D_\omega \varphi + \varphi \wedge \varphi = R(\omega) + [\![\omega,\varphi]\!] + R(\varphi) \tag{73}$$

with $R(\varphi) = d\varphi + \varphi \wedge \varphi$; the left decomposition above manifests covariance with respect to orthonormal transformations (because φ is a frame-space coordinate tensor), the right one manifests covariance with respect to internal-space transformations, see eq. (80) section 3.3. Finally, since $R(\omega)$ and $[\![\omega,\varphi]\!]$ are traceless in frame-space, we have:

$$\mathcal{R}^\alpha{}_\alpha(\varpi) \equiv R^\alpha{}_\alpha(\varphi) = d\varphi^\alpha{}_\alpha + \varphi^\beta{}_\alpha \wedge \varphi^\alpha{}_\beta. \tag{74}$$

This curvature-trace piece is known by the name *segmental curvature*.[14] We shall come back to it later in section 4 when we'll deal with actions.

In what follows, a post-Riemannian spacetime, equipped with a metric structure g, and with the connection (68), will be termed *merged spacetime*, and will be denoted by $(L_n\,[A]\,,g)$.

3.3 Invariance of the (encrusted) spacetime structural identities under internal gauge transformations

According to the prescription presented above, $\boldsymbol{q}_\alpha$ should possess an internal-space non-linear gauge transformation law (recall definition (62) and eq. (67)):

$$A\otimes\mathbf{q}_\alpha \;\;\mapsto\;\; uAu^{-1}\otimes\mathbf{q}_\alpha+udu^{-1}\otimes\boldsymbol{e}_\alpha. \tag{75}$$

Multiplying this by $\boldsymbol{e}^\beta$ gives:

$$\varphi^\beta{}_\alpha \;\;\mapsto\;\; u\varphi^\beta{}_\alpha u^{-1}+udu^{-1}\otimes\delta^\beta_\alpha \tag{76}$$

as required. Notice, however, that the off-diagonal components in $\varphi^\beta{}_\alpha$ transform *covariantly* with respect to internal gauge transformations. In terms of the quantities $Q^\beta{}_\alpha=-\varphi^\beta{}_\alpha$, and $Q_{\alpha\beta}=g_{\alpha\gamma}Q^\gamma{}_\beta$, the mapping in (76) reads:

$$Q^\beta{}_\alpha \;\;\mapsto\;\; uQ^\beta{}_\alpha u^{-1}-udu^{-1}\otimes\delta^\beta_\alpha \quad\Rightarrow\quad Q_{\alpha\beta}\;\;\mapsto\;\; uQ_{\alpha\beta}u^{-1}-udu^{-1}\otimes g_{\alpha\beta}, \tag{77}$$

hence the components of the non-metricity tensor transform as

$$Q_{(\alpha\beta)}\;\;\mapsto\;\; uQ_{(\alpha\beta)}u^{-1}-2udu^{-1}\otimes g_{\alpha\beta}. \tag{78}$$

[14] I thank Yuri Obukhov for notifying me on this matter; in ref. [11] the word *ric* was used instead. This shouldn't be confused with the so-called Ricci tensor, $\mathcal{R}_{\beta\gamma}:=\boldsymbol{e}_\gamma\rfloor\boldsymbol{e}_\alpha\rfloor\mathcal{R}^\alpha{}_\beta=\mathcal{R}^\alpha{}_{\beta\alpha\gamma}$, where a tensorial index is contracted with a basespace index. Here $\rfloor$ designates *interior multiplication*, see [4, Appendix A.1.3] for a summary of its properties; $\boldsymbol{v}\rfloor h$ is also said to be the *evaluation* of the form h over the vector field $\boldsymbol{v}$ [2].

In particular, in a *Weyl-Cartan merged spacetime* $(Y_n[A], g)$, where we have $Q^{\beta}{}_{\alpha} = -A \otimes \delta^{\beta}_{\alpha}$ (details are given in section 4.2), the left mapping in (77) translates into

$$A \mapsto uAu^{-1} + udu^{-1}, \tag{79}$$

whence A becomes a typical Yang-Mills (YM) gauge potential.

Under the mapping in (76), and since ω is indifferent to the internal world where A lives (so that it *gradely* commutes with udu^{-1}),

$$\left.\begin{array}{rcl} R(\omega) & \mapsto & R(\omega) \\ R(\varphi) & \mapsto & uR(\varphi)\,u^{-1} \\ [\![\omega,\varphi]\!] & \mapsto & u\,[\![\omega,\varphi]\!]\,u^{-1} \end{array}\right\} \overset{\text{by (73)}}{\Longrightarrow} \quad \mathcal{R}(\varpi) \mapsto u\mathcal{R}(\varpi)\,u^{-1}; \tag{80}$$

therefore, $\mathcal{R}^{i}{}_{j}(\varpi)$ transforms covariantly also with respect to internal gauge transformations. For example, the brackets term in (80) transforms as:

$$[\![\omega,\varphi]\!]^{\beta}{}_{\alpha} \mapsto u\,[\![\omega,\varphi]\!]^{\beta}{}_{\alpha}\,u^{-1} + \omega^{\gamma}{}_{\alpha} \wedge udu^{-1} \otimes \delta^{\beta}_{\gamma} + \left(udu^{-1} \otimes \delta^{\gamma}_{\alpha}\right) \wedge \omega^{\beta}{}_{\gamma} = u\,[\![\omega,\varphi]\!]^{\beta}{}_{\alpha}\,u^{-1}. \tag{81}$$

Two essential questions, however, immediately arise:

1. By allowing non-Abelian configurations in the base-part of the distortion 1-form, aren't we altering the form of the spacetime structural identities?

2. Assuming we don't, are these identities (which have now become "contaminated" by internal degrees of freedom) indifferent to internal gauge transformations?

We stress that the gauge invariance of the structural identities is an *indispensable* consistency requirement without which nothing makes sense; had the identities been sensitive to the gauge, then different gauges would associate with different structural equations.

In order to give an answer to the first question, we must re-derive the three identities, this time taking into account the fact that ϖ, Q, and T no longer gradely commute with each other (nor do they commute with themselves). A repeated careful computation of $d^2\vartheta^\alpha$ then explicitly reveals that the 1-st identity,

$$(D_\varpi T)^\alpha = \vartheta^\beta \wedge \mathcal{R}^\alpha{}_\beta\,(\varpi)\,, \tag{82}$$

remains intact while passing from Abelian to non-Abelian configurations.

As for the 2-nd Bianchi identity, we still identically have $D_\varpi \mathcal{R}\,(\varpi) = 0$ but, as opposed to eq. (53), this time

$$d\boldsymbol{\mathcal{R}} \;=\; \left(\mathcal{R}^\alpha{}_\gamma \wedge Q^\gamma{}_\beta - \mathcal{R}^\gamma{}_\beta \wedge Q^\alpha{}_\gamma\right) \boldsymbol{e}^\beta \otimes \boldsymbol{e}_\alpha \;\neq\; [\![\boldsymbol{\mathcal{R}}, \boldsymbol{Q}]\!]\,, \tag{83}$$

because $\boldsymbol{\mathcal{R}}$ and $\boldsymbol{Q}$ no longer commute (unless A is Abelian).

The 3-rd structural identity, $(D_\varpi Q)_{(\alpha\beta)} = -R_{(\alpha\beta)}$ (eq. (55)), nevertheless, requires more care in deriving it: one should be careful indeed not to make a slip on the ordering of non-commuting terms, especially in the expression for the covariant exterior derivative of $Q_{(\alpha\beta)}$. Doing so, we arrive at a 3-rd identity suitable for a merged spacetime:

$$\begin{aligned} 0 \;=\; ddg_{\alpha\beta} \;&=\; \left(\varpi^\delta{}_\gamma \wedge \varpi^\gamma{}_\alpha + \varpi^\gamma{}_\alpha \wedge \varpi^\delta{}_\gamma\right) g_{\delta\beta} \;+\; \left(\varpi^\delta{}_\gamma \wedge \varpi^\gamma{}_\beta + \varpi^\gamma{}_\beta \wedge \varpi^\delta{}_\gamma\right) g_{\delta\alpha} \\ &\quad+\; \left(\varpi^\gamma{}_\alpha \wedge \varpi^\delta{}_\beta + \varpi^\gamma{}_\beta \wedge \varpi^\delta{}_\alpha\right) g_{\gamma\delta} \;-\; \mathcal{R}_{(\alpha\beta)} \;-\; (D_\varpi Q)_{(\alpha\beta)} \end{aligned} \tag{84}$$

or, in a more compact and concise form,

$$-\,(D_\varpi \varphi)_{(\alpha\beta)} \;=\; [\![\varphi,\varphi]\!]_{(\alpha\beta)} + (\varphi \wedge \varphi)_{(\alpha\beta)} - \mathcal{R}_{(\alpha\beta)}. \tag{85}$$

For an Abelian connection (distortion), each parentheses in eq. (84) (eq. (85)) vanish, and we are left with the old relation, eq. (55). However, if φ is non-Abelian, due to a non-Abelian A, the parentheses terms remain, and eq. (55) does get corrected.

Let us now turn to the acute question of gauge-invariance. Recall first that $\mathcal{R}(\varpi)$ is U-covariant (eq. (80)). In particular, since ϖ transforms as a U-connection (eq. (69)), the covariant exterior derivative of the curvature 2-form is also U-covariant:

$$(D_{\varpi}\mathcal{R})^{\beta}{}_{\alpha} \;\mapsto\; u\,(D_{\varpi}\mathcal{R})^{\beta}{}_{\alpha}\,u^{-1}. \tag{86}$$

Hence, if $D_{\varpi}\mathcal{R}$ vanishes in one gauge (as it identically does), it will vanish in all gauges; the 2-nd identity is therefore insensitive to internal gauge transformations.

From the deformation criterion (eq. (60)), and from eq. (65), the encrusted torsion (in its coordinate vector form) reads: $T^{\alpha} = \varphi^{\alpha}{}_{\beta} \wedge \vartheta^{\beta}$. Invoking the internal space transformation rule for $\varphi^{\alpha}{}_{\beta}$ (given in (76)), reveals the way T^{α} transforms:

$$T^{\alpha} \;\mapsto\; uT^{\alpha}u^{-1} + udu^{-1} \wedge \vartheta^{\alpha}\,; \tag{87}$$

as a consequence,

$$\begin{aligned} dT^{\alpha} \;&\mapsto\; du \wedge T^{\alpha}u^{-1} + udT^{\alpha}u^{-1} + uT^{\alpha} \wedge du^{-1} + du \wedge du^{-1} \wedge \vartheta^{\alpha} - udu^{-1} \wedge d\vartheta^{\alpha}, \\ T^{\beta} \wedge \varpi^{\alpha}{}_{\beta} \;&\mapsto\; uT^{\beta} \wedge \varpi^{\alpha}{}_{\beta}u^{-1} + uT^{\alpha} \wedge du^{-1} - du \wedge \vartheta^{\beta} \wedge \varpi^{\alpha}{}_{\beta}u^{-1} + du \wedge du^{-1} \wedge \vartheta^{\alpha}. \end{aligned} \tag{88}$$

Substituting now (recall that $d\vartheta^{\alpha}$ gradely commutes with everything)

$$\begin{aligned} -\,du \wedge \vartheta^{\beta} \wedge \varpi^{\alpha}{}_{\beta}u^{-1} \;&=\; du \wedge d\vartheta^{\alpha}u^{-1} + du \wedge T^{\alpha}u^{-1} \\ &=\; -udu^{-1} \wedge d\vartheta^{\alpha} + du \wedge T^{\alpha}u^{-1} \end{aligned} \tag{89}$$

in the second line of (88), and subtracting it from the first line, cancels all non-linear terms in the transformed $(D_{\varpi}T)^{\alpha}$, leading eventually to a covariant behavior:

$$(D_{\varpi}T)^{\alpha} \;\mapsto\; u\,(D_{\varpi}T)^{\alpha}\,u^{-1}. \tag{90}$$

Thus, since the curvature as well transforms as an internal-space tensor (formula (80)), and since the coframe transforms as an internal-space scalar, the 1-st identity (eq. (82)) remains intact under internal gauge transformations.

We still have to check whether the upgraded 3-rd identity, eq. (84), is gauge-invariant. In fact, it is sufficient to check it only with respect to its trace in frame space, and this task is already not too cumbersome. First we write $g^{\alpha\beta}\left(D_{\varpi}Q\right)_{(\alpha\beta)} = 2dQ^{\alpha}{}_{\alpha} + Q^{(\alpha\beta)} \wedge Q_{(\alpha\beta)}$; then, since the non-Abelian piece in $\varpi_{\alpha\beta}$ is $\varphi_{\alpha\beta} = -Q_{\alpha\beta}$, and since $Q^{(\alpha\beta)} \wedge Q_{(\alpha\beta)}$ splits into $2Q^{\alpha\beta} \wedge Q_{\alpha\beta} + 2Q^{\alpha\beta} \wedge Q_{\beta\alpha}$, the trace of eq. (85) reduces to the obvious equation,

$$\mathcal{R}^{\alpha}{}_{\alpha} = d\varphi^{\alpha}{}_{\alpha} + \varphi^{\beta}{}_{\alpha} \wedge \varphi^{\alpha}{}_{\beta} := sc\left(\varphi\right) \tag{91}$$

($sc\left(\varphi\right)$ stands for *segmental curvature*, see eq. (74)) which manifests (internal-space) gauge invariance by virtue of the mapping rule for φ, formula (76).

3.4 Protecting the metric sector against invasions from the internal world

The absolute differential of a frame, and the exterior derivative of a coframe,

$$d\boldsymbol{e} = -\varpi\boldsymbol{e} - \boldsymbol{q}, \quad \text{and} \quad d\vartheta = -\vartheta \wedge \varpi - T, \tag{92}$$

are real-valued because they can always be brought into orthogonal form: $d\boldsymbol{e} = -\omega\boldsymbol{e}$ and $d\vartheta = -\vartheta \wedge \omega$, respectively, and ω is real-valued by default. The distortion 1-form $\boldsymbol{q}\left(A\right) \cdot \boldsymbol{e}^{-1}$ is surly complex-valued but only through its non-framed entrails, via A, and is therefore of no harm as far as the metric sector is concerned. Yet, the following claim merits a conduct: already at the level of the inclusion relations in (72), the elements in the subalgebra $\mathfrak{o}$, and those of the coset space $\mathfrak{s} = \mathfrak{l}/\mathfrak{o}$, mix under commutation relations. Furthermore, the hyper algebra $\left(\mathfrak{o} \otimes \mathbf{1}_{\rho(\mathfrak{u})}\right) \oplus \left(\mathfrak{s} \otimes \mathfrak{u}\right)$ is in general not a Lie algebra since its elements usually do not

close under brackets. Gauge transformations based on the exponentiation of this hyper algebra are therefore expected to stain the metric branch with internal degrees of freedom. One might thus (*wrongly*) infer that this already renders the merger concept non-realistic.

Surely, our theory can *not* be based on a hyper symmetry structure whose gauge group $\mathcal{G}$ is generated by the hyper algebra depicted in (70). Instead we base it on the *merging* of the underlying two symmetries, put in the form of a spacetime model. However, this can be established only if the transformations at the frame sector and those of the internal world remain independent of each other whatsoever (namely commute).

Let $\theta(x)$ be a local angle with values in the hyper algebra (70),

$$\theta \in \left[\mathfrak{o}(p,q) \otimes \mathbf{1}_{\rho(U)}\right] \oplus \left[\frac{\mathfrak{l}(n)}{\mathfrak{o}(p,q)} \otimes \mathfrak{u}(N)\right]; \tag{93}$$

let also the *Capital trace* $\mathrm{Tr}(*)$ designate a trace taken in the representation space of U, and the *lowercase trace* $\mathrm{tr}(*)$ designate a trace taken in frame space. Then, *assuming* that $\mathfrak{l}$ and $\mathfrak{u}$ each carrying a trace, we assign:

$$\ell(\theta) := \exp \mathrm{Tr}\, i\theta \in L(n,\mathbb{R}); \qquad u(\theta) := \exp \mathrm{tr}\, i\theta \in U(N,\mathbb{C}). \tag{94}$$

The assignments made here guarantee that frame transformations will never involve internal degrees of freedom, and that internal transformations will never interfere with the frames; furthermore, the two transformations in (94) are independent of each other. In this way the segregation between the metric and the non-metric sectors is preserved under any of these transformations despite the fact that in both cases the transformation is compensated by a *single common* connection ϖ, with values in the hyper algebra (70).

The true symmetry of the merger has thus been set to be $L(n) \times U(N)$, rather than the grand group $\mathcal{G}$, whose elements are given by $e^{i\theta}$. According to assignments (94), the presence

of trace pieces in $\mathfrak{u}$ enables shear and dilation transformations at the frames sector, whereas the dilation elements in $\mathfrak{l}$ give rise to internal gauge transformations. Otherwise, in the absence of trace pieces in $\mathfrak{u}$, the left-hand exponent in (94) reduces to local (pseudo-) rotations, while in the absence of dilation elements in $\mathfrak{l}$ (and since $\mathfrak{o}$ is already traceless in framespace) the right-hand exponent in (94) reduces to the identity element.

4 Merger of Yang-Mills interactions with gravity

4.1 Selecting an appropriate action

Our aim now is to select a suitable action for the dynamics of the merged spacetime, one that economically encompasses the two interlaced symmetries. There is probably no unique prescription for that task, but we surly must follow two basic requirements:

1. The action must be invariant under frame transformations;

2. It must also be invariant under internal gauge transformations.

With these guidelines in mind, and knowing in advance the outcome of our choice, we proceed along the lines of [11] and choose a rather minimal option:[15]

$$S_{LC}\left(g,\omega,\varphi\right) \;=\; \int_{\Omega} \mathrm{Tr}\left[\mathcal{R}_{\alpha\beta}\wedge\star\left(\frac{1}{\ell^{n-2}}\vartheta^{\alpha}\wedge\vartheta^{\beta}+\frac{1}{\ell^{n-4}}g^{\alpha\beta}\mathcal{R}^{\gamma}{}_{\gamma}\right)\right]; \tag{95}$$

here $\star$ stands for the Hodge-star map, the parameter ℓ, having the dimension of length, has been installed with the prescribed powers, in order to make the two term in the integrand dimensionless[16], and the capital trace is taken in the representation space of U. The $\star\,(*)$

[15] This type of action, and some variants alike, have also been introduced in [12].

[16] This is, of course, the origin of the gravitational constant in $n = 4$.

term in (95) should be contemplated as a generalized *excitation tensor* [4]; being adjusted to post-Riemannian geometry, it now contains a *symmetric* fiber piece, $g^{\alpha\beta} = \boldsymbol{e}^{\alpha} \cdot \boldsymbol{e}^{\beta}$, in addition to the traditional anti-symmetric one, $\vartheta^{\alpha} \wedge \vartheta^{\beta}$.

The action functional (95) decomposes into two terms,

$$S_{LC}\left(g, \omega, \varphi\right) \;=\; \frac{1}{\ell^{n-2}} \int_{\Omega} \mathrm{Tr}\left[\mathcal{R}_{\alpha\beta}\left(\omega, \varphi\right) \wedge \star\left(\vartheta^{\alpha} \wedge \vartheta^{\beta}\right)\right] + \frac{1}{\ell^{n-4}} \int_{\Omega} \mathrm{Tr}\left[sc\left(\varphi\right) \wedge \star\, sc\left(\varphi\right)\right], \tag{96}$$

the first of which *resembles* the gravitational Einstein-Hilbert (EH) action in the absence of matter sources, and will thus be identified with the gravitational branch of the merger. The second integral, on the other hand, will be associated with internal space dynamics; in one particular case it will eventually generate a YM source for energy-momentum in Einstein's gravitational theory.

Clearly, we could have added an additional $\mathcal{R}^{\alpha\beta}$ term into the excitation tensor in (95). A curvature square term in the action would have represent contributions that dominate at high curvature,[17] and may possibly become relevant deep in the quantum regime. Apart from the Weyl-Cartan case, it will produce strong coupling between ω-dependent and φ-dependent terms; in any situation, it will also give rise to the presence of the norm,

$$\int_{\Omega} R^{\alpha}{}_{\beta}\left(\omega\right) \wedge \star R^{\beta}{}_{\alpha}\left(\omega\right) \tag{97}$$

namely, to the existence of a YM-type term for the gravitational field, in addition to the EH term. Such high-curvature corrections (even if they can be justified on theoretical grounds) will not be discussed in the sequel.

[17] The absence of this term therefore characterizes the low-curvature (or classical) limit of the theory, hence the subscript "LC" attached to $S_{LC}\left(\omega, \varphi\right)$ in (95)-(96).

4.2 The Weyl-Cartan merger: structure, features, and dynamics

The Weyl-Cartan spacetime is the simplest extension of a Riemannian spacetime that goes beyond the constraints of metricity and holonomicity. It possesses post-Riemannian geometry with non-vanishing torsion and non-metricity generated by a minimal number of non-metric degrees of freedom. Our objective in this section is to show explicitly that the Weyl-Cartan *merged* spacetime $(Y_n[A], g)$ provides a comprehensive (classical) framework for the description of the (bosonic sector of the) present days low energy physics.

In the Weyl-Cartan spacetime, the vector-valued 0-form $\mathfrak{q}_\alpha$ (defined in eq. (62)) is taken to be equal to the frame field $\boldsymbol{e}_\alpha$ itself,

$$\mathfrak{q}_\alpha \equiv \boldsymbol{e}_\alpha\,; \quad \text{hence, from eqs. (62) and (65)}: \quad \mathcal{Q}^\beta{}_\alpha = \delta^\beta_\alpha \;\Rightarrow\; \varphi^\beta{}_\alpha = A \otimes \delta^\beta_\alpha. \tag{98}$$

It then follows that the coset space $\mathfrak{s}\,(= \mathfrak{l}/\mathfrak{o})$ in association (71) contains only the identity element in frame space; ϖ takes its values $\in \left[\mathfrak{o} \otimes \mathbf{1}_{\rho(U)}\right] \oplus \left[\mathbf{1}_{\rho(O)} \otimes \mathfrak{u}\right]$, the Lie algebra of the direct product group $O \times U$, and therefore has the algebraic structure of a connection in a Whitney product of two vector bundles [10].

The assignment in (98) leads to particularly simple expressions for the torsion and non-metricity (eqs. (63) and (64), respectively):

$$T^\alpha \;=\; A \wedge \vartheta^\alpha, \qquad Q_{(\alpha\beta)} \;=\; -2A \otimes g_{\alpha\beta}\,; \tag{99}$$

it then follows that transvection terms of the form: $\vartheta_\alpha \wedge T^\alpha$, $T_\alpha \wedge T^\alpha$, and $Q_{(\alpha\beta)} \wedge T^\alpha \wedge \vartheta^\beta$ (which we discuss in the Appendix), identically vanish.

The segmental curvature, eq. (74), now takes the form of a YM curvature,

$$sc(\varphi) \;=\; n\,(dA + A \wedge A) \;=:\; nF(A)\,, \tag{100}$$

in which case the integrand in the second term of $S_{LC}(g,\omega,\varphi)$ in (96) makes up a free YM Lagrangian for the potential A,

$$\mathrm{Tr}\,(sc\wedge\star sc)=n^2\,\mathrm{Tr}\,(\,F\wedge\star F)\,. \tag{101}$$

By formula (77), and as has already been stated in (79), A and F satisfy the YM gauge transformation laws,

$$A\ \ \mapsto\ \ uAu^{-1}+udu^{-1},\qquad F\ \ \mapsto\ \ uFu^{-1}. \tag{102}$$

Since the internal branch of the merger acquires the exact content of a classical YM gauge theory, we may now explicitly assign:

$$A^i{}_j\ :=\ A^a_\alpha\,(I_a)^i{}_j\,\vartheta^\alpha\ \ \Rightarrow\ \ F^i_{\ j}\ =\ \tfrac{1}{2}F^a_{[\alpha\beta]}\,(I_a)^i{}_j\,\vartheta^\alpha\wedge\vartheta^\beta, \tag{103}$$

where the I_a's stand for the generators of U.[18]

From $\mathcal{Q}^\beta{}_\alpha\ =\ \delta^\beta_\alpha$, and because A gradely commutes with ω, the brackets $[\![\omega,\varphi]\!]$ in (73) identically vanish, and the curvature of the Weyl-Cartan merger decomposes as:

$$\mathcal{R}^\beta{}_\alpha\,(\varpi)\ =\ R^\beta{}_\alpha\,(\omega)\otimes\mathbf{1}_{\rho(U)}+F\,(A)\otimes\delta^\beta_\alpha, \tag{104}$$

or in its "lowercase" form: $\mathcal{R}_{\alpha\beta}\,(\varpi)=R_{\alpha\beta}\,(\omega)\otimes\mathbf{1}_{\rho(U)}+F\,(A)\otimes g_{\alpha\beta}$.[19] Decomposition (104) can also be viewed as if the Riemannian curvature went through - what we call - *projective*

[18] Hence, if $\mathcal{U}_p$ is a frame-space scalar p-form, transforming as the components of a tensor under the action of U, namely $\forall\ u\in U:\ \mathcal{U}_p\ \mapsto u\mathcal{U}_pu^{-1}$, then, according to (102), the covariant exterior derivative of $\mathcal{U}_p$ with respect to the connection A is given by

$$\begin{aligned}(D_A\mathcal{U})_{p+1}\ &:=\ d\mathcal{U}_p+A\wedge\mathcal{U}_p+(-1)^{p+1}\,\mathcal{U}_p\wedge A=d\mathcal{U}_p+[\![A,\mathcal{U}_p]\!]\,,\\ \text{and}\ \ (D_AD_A\mathcal{U})_{p+2}\ &=\ [F\,(A)\,,\mathcal{U}_p]\,.\end{aligned}$$

[19] Clearly, since $\varpi\in\left[\mathfrak{o}\otimes\mathbf{1}_{\rho(U)}\right]\oplus\left[\mathbf{1}_{\rho(O)}\otimes\mathfrak{u}\right]$, it follows that $\mathcal{R}\,(\varpi)\in\left[\mathfrak{o}\otimes\mathbf{1}_{\rho(U)}\right]\oplus\left[\mathbf{1}_{\rho(O)}\otimes\mathfrak{u}\right]$ as well.

non-Abelian deformation: from an internal space perspective, since $F(A)$ identically satisfies the Bianchi identity with respect to A (see footnote 18),

$$D_A F(A) \;=\; dF(A) + A \wedge F(A) - F(A) \wedge A \;=\; 0, \tag{105}$$

the deformation term in decomposition (104) closes with respect to D_A.

For an Abelian gauge field, the deformation term in eq. (104), $F(A) \otimes \mathbf{1}_n \;=\; dA \otimes \mathbf{1}_n$, becomes a proper projective element [4, eq. (3.11.8)]. In particular, in four dimensions, and after renaming the gauge field, $A \to -\frac{1}{4}A$, one encounters the triplet

$$\boldsymbol{Q} \;=\; \frac{1}{2}A \otimes \boldsymbol{g}, \qquad \boldsymbol{T} \;=\; -\frac{1}{4}\left(A \wedge \vartheta^\alpha\right) \otimes \boldsymbol{e}_\alpha, \qquad sc(A) \;=\; -dA, \tag{106}$$

which is nothing but the Teyssandier-Tucker vacuum configuration, discovered in '96 [11]; in their seminal paper Teyssandier and Tucker proposed the action (96) (in four dimensions, with an Abelian gauge field A, and equipped with a coupling constant also in front of the second term) and looked for a configuration that *extremes* it.

In addition to identity (105), and due to $D_\varpi \mathcal{R}(\varpi) = 0$, we have one more curvature identity, namely the 2-nd Bianchi identity associated with the rotational subgroup,

$$D_\omega R(\omega) = 0, \tag{107}$$

and essentially *two* more torsional identities:

$$\begin{aligned}
(D_\varpi T)^\alpha \;&=\; d\left(A \wedge \vartheta^\alpha\right) - \left(A \wedge \vartheta^\beta\right) \wedge \varpi^\alpha{}_\beta \;=\; dA \wedge \vartheta^\alpha - A \wedge \left(d\vartheta^\alpha + \vartheta^\beta \wedge \varpi^\alpha{}_\beta\right) \\
&=\; dA \wedge \vartheta^\alpha + A \wedge T^\alpha \;=\; F(A) \wedge \vartheta^\alpha,
\end{aligned} \tag{108}$$

whence, due to eqs. (82) and (104),

$$\vartheta^\beta \wedge R^\alpha{}_\beta(\omega) \;=\; 0. \tag{109}$$

From a geometric (and aesthetic) viewpoint, the interrelations between the internal gauge field A and the geometry of the $(Y_n[A], g)$ spacetime may concisely be summarized in a triplet of pairs of equivalent objects:

$$-A \otimes \boldsymbol{g} \;\leftrightarrow\; \boldsymbol{Q}, \qquad A \wedge \boldsymbol{\pi} \;\leftrightarrow\; \boldsymbol{T}, \;\text{ and }\; F(A) \wedge \boldsymbol{\pi} \;\leftrightarrow\; D_{\varpi}\boldsymbol{T} \tag{110}$$

(recall that $\boldsymbol{\pi} := \vartheta^{\alpha}\boldsymbol{e}_{\alpha}$); the YM gauge field can therefore be interpreted as a non-Abelian source for torsion and non-metricity in a (Y_n, g) spacetime.[20]

Let us next compute $(D_{\varpi}Q)_{(\alpha\beta)}$. Since $Q_{(\alpha\beta)} = -2A \otimes g_{\alpha\beta}$, and from the relation $dg_{\alpha\beta} = -\varpi_{\alpha\beta} - \varpi_{\beta\alpha} - Q_{(\alpha\beta)}$ we find:

$$\begin{aligned}(D_{\varpi}Q)_{(\alpha\beta)} &= -2\left(dA \otimes g_{\alpha\beta} - A \wedge dg_{\alpha\beta} - A \wedge \varpi_{\alpha\beta} - A \wedge \varpi_{\beta\alpha}\right) \\ &= -2\left(dA \otimes g_{\alpha\beta} + A \wedge Q_{(\alpha\beta)}\right) \;=\; -2F(A) \otimes g_{\alpha\beta} + 6A \wedge A \otimes g_{\alpha\beta},\end{aligned} \tag{111}$$

which is seen to hold in agreement with the 3-rd identity, eq. (84). Transvecting this with $g^{\alpha\beta}$ (namely, taking the trace in frame-space) gives,

$$g^{\alpha\beta}(D_{\varpi}Q)_{(\alpha\beta)} = -2n\left[F(A) - 3A \wedge A\right]; \quad \text{also,} \quad g^{\alpha\beta}Q_{(\alpha\beta)} = -2nA. \tag{112}$$

These two may now be substituted into the formula for the ($(L_n[A], g)$-upgraded) *non-Abelian Stokes' theorem* for the non-metricity-trace,

$$\int_{\Sigma} g^{\alpha\beta}(D_{\varpi}Q)_{(\alpha\beta)} - \int_{\Sigma} Q^{(\alpha\beta)} \wedge Q_{(\alpha\beta)} \;=\; \oint_{\partial\Sigma} g^{\alpha\beta}Q_{(\alpha\beta)}, \tag{113}$$

[20] In ref. [13] (bottom of page 9) we found a stinging remark saying, inter alia, that the torsion cannot be closely related to electromagnetism, nor to any other non-gravitational (gauge) field. This, however, seems to be true only as long as one considers the *vectorial* component of the torsion. Nothing negates close relations between the basespace components of the torsion and the gauge fields of an internal symmetry.

which we develop in Appendix A.2 (here Σ stands for a 2-domain that can be covered by a *single* patch), to yield the (obvious) result,

$$\int_{\Sigma} dA = \oint_{\partial\Sigma} A; \tag{114}$$

one may consider this result as a nice consistency checking.

Since $\mathcal{R}_{\alpha\beta}(\varphi,\omega) = R_{\alpha\beta}(\omega) \otimes \mathbf{1}_{\rho(U)} + F(A) \otimes g_{\alpha\beta}$, and due to $g_{\alpha\beta} \star \left(\vartheta^{\alpha} \wedge \vartheta^{\beta}\right) \equiv 0$, the gravitational term in the r.h.s. of (96) reduces to the EH action of General Relativity (GR); on the other hand, the internal term reduces to free YM (eq. (100)). Hence, our proposed $(Y_n[A], g)$ action takes the compelling form

$$S_{LC}(g,\omega,A) = \frac{\dim\rho}{\ell^{n-2}} \int_{\Omega} R_{\alpha\beta}(\omega) \wedge \star\left(\vartheta^{\alpha} \wedge \vartheta^{\beta}\right) + \frac{n^2}{\ell^{n-4}} \int_{\Omega} \mathrm{Tr}\left[F(A) \wedge \star F(A)\right], \tag{115}$$

with $R(\omega) = d\omega + \omega \wedge \omega$, and $F(A) = dA + A \wedge A = (D_{\omega}A)_{\alpha} \wedge \vartheta^{\alpha} + A \wedge A$.[21] Needless to elaborate, the action (115) describes a (non-Abelian) YM gauge field in the presence of Einstein's gravity in n dimensions.

The equations of motion that come out by extremising $S_{LC}(g,\omega,A) = S_{LC}(\vartheta,\omega,A)$ with respect to $g_{\alpha\beta}$ are known to treat the YM curvature as an internal-space source for energy momentum in Einstein's equation; this is a standard procedure. Alternatively, we may vary the action in (115) with respect to the *coframes* to get the very same result.[22] The latter method, however, reveals interesting dynamical features in a transparent manner, hence we shall now work it out in details. Since $R(\omega)$ and $F(A)$ *do not depend on the choice of a coframe*, what we only need is a prescription for the variation of forms mapped under the Hodge star. Such prescription has recently been developed by Muench, Gronwald, and Hehl

[21] Clearly, in a holonomic basis $(D_{\omega}A)_{\alpha} \wedge \vartheta^{\alpha}$ reduces to $dA_{\alpha} \wedge \vartheta^{\alpha}$ due to $d\vartheta^{\alpha} = 0$.

[22] I thank Nico Giulini for explaining me this method of obtaining the Einstein equation.

[15, pages 8-9]: let ϕ be an arbitrary p-form (not carrying values in a Lie algebra). Then,

$$\begin{aligned}\delta \star \phi &= \star\, \delta\phi + \delta\vartheta^\alpha \wedge (\boldsymbol{e}_\alpha \rfloor \star \phi) - \star\left[\delta\vartheta^\alpha \wedge (\boldsymbol{e}_\alpha \rfloor \phi)\right] \\ &\quad + \delta g_{\alpha\beta}\left[\vartheta^{(\alpha} \wedge \left(\boldsymbol{e}^{\beta)} \rfloor \star \phi\right) - \frac{1}{2} g^{\alpha\beta} \star \phi\right], \end{aligned} \tag{116}$$

where $\rfloor$ denotes interior multiplication (see footnote (14)).

We shall now vary the *coframes* in (115), leaving all other fields in the action - including the components of the metric tensor - untouched (so that $\delta g_{\alpha\beta} = 0$). Using $\mathrm{Tr}\,(F \wedge \star F) = \sum_a F^a \wedge \star F^a$ with $a = 1 \cdots \dim U$, the evaluation formula $\boldsymbol{e}_\alpha \rfloor \vartheta^\beta = \delta^\beta_\alpha \Rightarrow \boldsymbol{e}^\alpha \rfloor \vartheta^\beta = g^{\alpha\beta}$, and the identity $\int_\Omega \sigma \wedge \star \phi = \int_\Omega \phi \wedge \star \sigma$, which holds for any two scalar-valued forms σ and ϕ of equal degree, we obtain:

$$\delta_\vartheta \int_\Omega R_{\alpha\beta} \wedge \star \left(\vartheta^\alpha \wedge \vartheta^\beta\right) = \int_\Omega \delta\vartheta^\gamma \wedge R_{\alpha\beta} \wedge \boldsymbol{e}_\gamma \rfloor \star \left(\vartheta^\alpha \wedge \vartheta^\beta\right), \tag{117}$$

$$\delta_\vartheta \int_\Omega \mathrm{Tr}\,(F \wedge \star F) = \int_\Omega \delta\vartheta^\gamma \wedge \mathrm{Tr}\left[F \wedge (\boldsymbol{e}_\gamma \rfloor \star F) - (\boldsymbol{e}_\gamma \rfloor F) \wedge \star F\right]. \tag{118}$$

Hence, as the variation $\delta\vartheta^\gamma$ is arbitrary, and since $\boldsymbol{e}_\alpha \rfloor \star \sigma = \star(\sigma \wedge \vartheta_\alpha)$, we finally arrive at the required equation(s) of motion:

$$R_{\alpha\beta} \wedge \star \left(\vartheta^\alpha \wedge \vartheta^\beta \wedge \vartheta^\gamma\right) = -\kappa\, \mathrm{Tr}\left[F \wedge \star (F \wedge \vartheta^\gamma) - (\boldsymbol{e}^\gamma \rfloor F) \wedge \star F\right], \tag{119}$$

with the constant $\kappa = n^2 \ell^2 / \mathrm{dim}\rho$ having the dimension of length-square. The l.h.s. of (119) is the *Einstein* $(n-1)$*-form* (see [16] for more details on this object); the r.h.s. contains the *energy-momentum current* associated with the YM gauge field,

$$\Sigma^\gamma := \mathrm{Tr}\left[F \wedge \star (F \wedge \vartheta^\gamma) - (\boldsymbol{e}^\gamma \rfloor F) \wedge \star F\right]. \tag{120}$$

As $\vartheta^\alpha \wedge (\boldsymbol{e}_\alpha \rfloor \sigma) = p\sigma$ for any p-form σ, the YM energy-momentum current satisfies:

$$\vartheta^\gamma \wedge \Sigma_\gamma = (n-4)\, \mathrm{Tr}\,(F \wedge \star F), \tag{121}$$

and it vanishes *only at* $n = 4$. The vanishing of $\vartheta^{\gamma} \wedge \Sigma_{\gamma}$ in four dimensions corresponds to the tracelessness of the standard energy momentum tensor in Einstein's gravity - see in this respect the Abelian treatment of ref. [17].

Taking the variation of $\mathcal{L}_{LC}(\vartheta, \omega, A)$ with respect to the YM gauge field is a standard procedure. The result (in the absence of spinorial matter or any external currents) is given by the so-called YM *vacuum* equation,

$$D_A \star F = 0. \tag{122}$$

In four dimensions the curvature eigenforms of the Hodge star map, $\star F = \alpha F$, where α is any scalar, are obvious solutions of eq. (122) due to the Bianchi identity for F, $D_A F = 0$. It is interesting to note that these special solutions correspond to non-trivial *gravitational emptiness* since, as can be vividly seen from the integrand in the r.h.s. of (118), they imply the vanishing of the energy-momentum current, $\Sigma^{\gamma} = 0$, hence reducing eq. (119) to the empty space equation, $R_{\alpha\beta} \wedge \star \left(\vartheta^{\alpha} \wedge \vartheta^{\beta} \wedge \vartheta^{\gamma}\right) = 0$. We therefore conclude that gauge field configurations which are "self-dual" solutions of the YM vacuum equation exert no influence on the (*Riemannian*) spacetime in which they dwell.

Finally, varying $\mathcal{L}_{LC}(\vartheta, \omega, A)$ with respect to the Riemannian connection ω reproduces the Riemannian relation $(D_{\omega}\vartheta)^{\alpha} = 0$ (see [14, eqs. (2.2)-(2.4)] for the detailed derivation) and therefore provides us with no further information. Hence eqs. (119) and (122) alone capture the entire dynamics of the $(Y_n[A], g)$ merger.

4.3 The Post-Riemannian action in a fixed non-metric gauge

In the more general case, $[\![\omega, \varphi]\!] \neq 0$, and $\mathcal{R}_{[\alpha\beta]}$ doesn't necessarily depend on ω solely. Using formula (73), and after making some rearrangements, we arrive at the following expression for

the Lagrangian at the gravitational sector:[23]

$$\begin{aligned}\mathcal{L}_{GS} &= \mathrm{Tr}\left[R_{\alpha\beta}(\omega)\otimes\mathbf{1}_{\rho(U)} + [\![\omega,\varphi]\!]_{\alpha\beta} + R_{\alpha\beta}(\varphi)\right]\wedge\star\left(\vartheta^{\alpha}\wedge\vartheta^{\beta}\right)\\ &= \mathrm{Tr}\left[R_{\alpha\beta}(\omega)\otimes\mathbf{1}_{\rho(U)} - A\wedge\left(D_{\omega}\mathcal{Q}\right)_{\alpha\beta} + dA\otimes\mathcal{Q}_{\alpha\beta}\right]\wedge\star\left(\vartheta^{\alpha}\wedge\vartheta^{\beta}\right).\end{aligned}\tag{123}$$

Clearly, if A takes values in a traceless algebra, the A-dependent terms disappear, and the action for the gravitational sector reduces to the ordinary EH action of GR. Otherwise, if A carries a trace, the gravitational Lagrangian contains new mixing terms in which the YM potential couples to the gravitational field.

In order to make the picture in this genuine case more transparent we pick a direction for $\mathfrak{q}_{\alpha}$, and rewrite the action in that gauge. This can be consistently done if we assume that it is possible to foliate each spacetime patch into slices of $(n-1)$-hypersurfaces. We may then align $\mathfrak{q}_{\alpha}$, in each of these patches, in the direction normal to the hypersurfaces, say along $\boldsymbol{e}_0$, and assign:

$$\mathfrak{q}_{\alpha} =: \frac{1}{n}Y(x)\,\delta^{0}_{\alpha}\boldsymbol{e}_0,\tag{124}$$

with an arbitrary *scaling function* $Y(x)$.[24] Note that by making this choice we haven't fallen back into a $(Y_n[A], g)$ because now $\mathfrak{q}_{\alpha}\neq\boldsymbol{e}_{\alpha}$.

With this alignment, and from eqs. (62), (63), and (64), the components of the distortion, the non-metricity, and the torsion read:

$$Q^{\beta}{}_{\alpha} = -\frac{1}{n}A\otimes Y\delta^{\beta}_{0}\delta^{0}_{\alpha}\,,\tag{125}$$

$$Q_{(\alpha\beta)} = -\frac{1}{n}A\otimes Y\left(\delta^{0}_{\alpha}g_{0\beta}+\delta^{0}_{\beta}g_{0\alpha}\right),\tag{126}$$

$$T^{\alpha} = -\frac{1}{n}\vartheta^{0}\wedge A\otimes Y\delta^{\alpha}_{0}\,,\tag{127}$$

[23] $\mathcal{R}_{\alpha\beta}(\varpi)\wedge\star\left(\vartheta^{\alpha}\wedge\vartheta^{\beta}\right)$ contains also the term $A\wedge A\otimes\mathcal{Q}^{\gamma}{}_{\beta}\mathcal{Q}_{\alpha\gamma}$ but its capital trace vanishes identically.

[24] Y can be interpreted as a local mussure for the lack of metricity; a vanishing Y corresponds to metricity.

or, in a matrix form, and in terms of the *scale-dependent* effective gauge field $A_Y := AY$,

$$Q_{()} = -\frac{1}{n} A_Y \otimes \left(\begin{array}{c|ccc} 2g_{00} & g_{01} & \cdots & g_{0n} \\ \hline g_{10} & & & \\ \vdots & & 0 & \\ g_{n0} & & & \end{array} \right), \qquad T = -\frac{1}{n} \vartheta^0 \wedge A_Y \otimes \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{128}$$

Then, the segmental curvature (eq. (91)) acquires a particularly simple structure:

$$\begin{aligned} sc\,(\varphi) &= d\,(A \otimes \mathcal{Q}^{\alpha}{}_{\alpha}) + (A \otimes \mathcal{Q}^{\gamma}{}_{\alpha}) \wedge \left(A \otimes \mathcal{Q}^{\alpha}{}_{\gamma}\right) \\ &= \frac{1}{n} \left(dA_Y + \frac{1}{n} A_Y \wedge A_Y \right) =: \frac{1}{n} F_{1/n}\,(A_Y)\,, \end{aligned} \tag{129}$$

namely, it is simply the YM curvature for the scale-dependent gauge field A_Y (with a meaningless prefactor $1/n$). From eqs. (76) and (125), under the gauge,

$$A_Y \otimes \delta^0_\alpha \delta^\beta_0 \;\mapsto\; u A_Y u^{-1} \otimes \delta^0_\alpha \delta^\beta_0 + n u du^{-1} \otimes \delta^\beta_\alpha\,; \tag{130}$$

therefore, that single element in $Q^{\beta}{}_{\alpha}$ which is not a pure gauge,[25] namely its 00-component given by A_Y, transforms as:

$$A_Y \;\mapsto\; u A_Y u^{-1} + n u du^{-1}, \quad \text{whence} \quad F_{1/n}\,(A_Y) \;\mapsto\; u F_{1/n}\,(A_Y)\, u^{-1}. \tag{131}$$

Written in the alignment (124) the extended EH action $S_{LC}\,(\vartheta, \omega, \varphi)$ given by (96) takes the compelling form:

$$\begin{aligned} S^{gf}_{LC}\,(\vartheta, \omega, A_Y) \;=\; & \frac{\dim \rho}{\ell^{n-2}} \int_\Omega R_{\alpha\beta}\,(\omega) \wedge \star \left(\vartheta^\alpha \wedge \vartheta^\beta\right) \;+\; \frac{1}{\ell^{n-4} n^2} \int_\Omega \mathrm{Tr} \left[F_{1/n}\,(A_Y) \wedge \star F_{1/n}\,(A_Y)\right] \\ & + \frac{1}{n \ell^{n-2}} \int_\Omega \mathrm{Tr} \left[F_{1/n}\,(A_Y)\right] \wedge \star \left(\vartheta_0 \wedge \vartheta^0\right) \\ & + \frac{1}{n \ell^{n-2}} \int_\Omega \mathrm{Tr} \left[A_Y\right] \wedge \left[\omega_{\alpha 0} \wedge \star \left(\vartheta^\alpha \wedge \vartheta^0\right) - \omega^{0\alpha} \wedge \star \left(\vartheta_0 \wedge \vartheta_\alpha\right)\right]. \end{aligned} \tag{132}$$

[25] According to (130) the null entries in $\mathrm{diag}\,(\varphi)$ are indeed null only up to a pure gauge, $A_{PG} \sim n u du^{-1}$.

Note that the last term in (132) is not preserved by internal gauge transformations, and it actually describes an interaction written in a fixed internal gauge. In fact, as one may directly infer from the detailed derivation of (132), aligning the deformation element $\boldsymbol{q}_\alpha$ in a certain direction (which is the same as to fix the direction of the torsion and non-metricity), amounts to *fixing a gauge* also in the internal space for those cases where A carries a trace.[26]

If A_Y and $F_{1/n}(A_Y)$ are traceless objects (in which case the symmetry on the frames is restricted to local pseudo-rotations, see section 3.4), the two pieces in $S_{LC}(\vartheta, \omega, A_Y)$ with the prefactor $1/n\ell^{n-2}$ in front drop out. Then the gauge invariance of the action is retained, and the internal world, at least at the level of the action, decouples from the gravitational world. The corresponding e.o.m.'s are those given by eqs. (119) and (122), with A being replaced by the scale-dependent field A_Y, $F(A)$ being replaced by $F_{1/n}(A_Y)$, and a different numerical prefactor, $\kappa' = \kappa/n^4$.

Otherwise, if A_Y has a non-vanishing trace, its trace part (or its "photonic" component) interacts with the gravitational field ω via the contact terms $\mathrm{Tr}(A_Y) \wedge \omega$. This interaction, as well as the 'vortex' term $\mathrm{Tr}\left(F_{1/n}\right)$, are of weak magnitude, namely $1/(n\,\mathrm{dim}\rho)$ times the gravitational magnitude. It is interesting to note that under the restriction to orthonormal frames, the two contact terms, and the vortex term, extinguish even for an *extrinsic* YM potential that has been placed by hand: in this case, since we have already been employing an-holonomic basis, $g^{\alpha\beta} = \eta^{\alpha\beta}$, $\eta^{\alpha 0} = \eta_{\alpha 0} = \pm\delta_{\alpha 0}$, whence

$$\eta_{\alpha 0}\vartheta^\alpha \wedge \vartheta^0 \;=\; \pm\vartheta^0 \wedge \vartheta^0 \;=\; 0, \quad \text{and} \quad \omega^{0\alpha} \wedge \star(\vartheta_0 \wedge \vartheta_\alpha) \;=\; \omega_{\alpha 0} \wedge \star(\vartheta^\alpha \wedge \vartheta^0)\,, \tag{133}$$

so that the terms with the prefactor $1/n\ell^{n-2}$ in (132) drop-out. Hence the new interactions cannot be detected in a V_n whatsoever.

[26]Hence the superscript "gf" in $S_{LC}^{gf}(\vartheta, \omega, A_Y)$.

It is admitted that the dynamics of the scaling function Y has been completely absorbed in the dynamics of the redefined potential A_Y. But: as $Y(x)$ encodes the microstructure of spacetime, different spacetime scales display different gluon structures, and this, in turn, may affect the phenomenology. For example, in the case of a traceless A, and for extremely small values of Y (compared to A), or when Y heavily fluctuates such that Y^2 becomes negligible compared to dY, we have: $F_{1/n}(A_Y) \approx dA_Y$, whence

$$L_{YM} \approx \mathrm{Tr}\left(dA_Y \wedge \star dA_Y\right), \tag{134}$$

and the internal (non-Abelian) world becomes effectively free.

5 Summary and closing remarks

The local texture of the frame bundle as a space product between the tangent frame and the base enables the *merging* of gravity with YM theory under the roof of post-Riemannian geometry. The spacetime geometry, which stands on three structural identities, and on the gauge symmetry induced on the frames, is not affected by internal gauge transformations, despite that the structural identities in this framework are loaded with internal degrees of freedom; furthermore, its metric part remains real-valued even for complex-valued internal symmetry groups. A post-Riemannian spacetime endowed with an internal symmetry was termed *merged spacetime* because the corresponding two symmetry structures, frames and internal, were merged at the gauge sector level into a single spacetime fabric.

From a tangent bundle perspective, the internal symmetry lies entirely on the basespace, whereas gravity emerges as the local symmetry structure associated with the frames. In this sense, a merged spacetime is a bundle setup where the fiber *and* the base play an active role. Algebraically speaking, excluding the Weyl-Cartan case, the overall gauge structure is not a

Whitney product of two vector bundles since the generating algebra is not a simple sum of algebras. From the physical perspective, *the torsion and non-metricity in spacetime, and the internal-space potential, are one and the same thing in their basespace components*.

The merging of gravity with YM interactions could not have been correctly established without referring explicitly to the post-Riemannian extension of GR. Had we originally split the Riemannian connection into its base-part and fiber-part, and attributed internal degrees of freedom to the former, we would have necessarily contaminated the frames, the coframes, and the metric tensor (which we had to symmetrize a-priori) by these degrees of freedom. In this case, not only we would have lost commutativity (the frames and coframes would have become non-Abelian), but the gravitational sector and the internal symmetry sector could not have been separated and split apart. Furthermore, since gravity is real-valued, any symmetry that dwells at the metric sector must be real-valued as well.

A motivated (and rather minimal) choice of action, based on an extended variant of the EH action, led to a natural split between the gravitational branch and the internal branch of the merger. In the Weyl-Cartan scenario, the gravitational branch is seen to consists of pure Riemannian gravity, while the internal branch displays the structure of pure YM, with its field strength building up the energy momentum current in Einstein's equation. It was shown that the energy momentum current is traceless only in four dimensions, and that the star eigenform curvature solutions to the YM field equations (in four dimensions) lead to an empty space Einstein equation. We therefore conclude that a self-dual YM potential has no dynamical effect on the Riemannian spacetime in which it dwells.

In the more general case, in a fixed distortion gauge, the gauge fields effectively become scale-dependent, and consequently sensitive to the non-metric microstructure of spacetime.

A glance at the detailed structure of the gauge-fixed action reveals that the YM potential weakly interacts with the gravitational potential via the “photonic” components it carries. Furthermore, the scaling function attributed to the gluons is seen to determine qualitatively what type of effective theory shows at the various situations; in particular, an effectively free theory is anticipated at weak, or at a heavily fluctuating non-metricity.

Our formalism obviates the need to introduce a symmetry-breaking mechanism for the post-Riemannian GR (so as to comply with the equivalence principle). Such mechanism was advocated in [18], and later elaborated and summarized in [4]. The trigger that ignites the symmetry-breaking in these studies is a dilaton field that originates from conformal symmetry added to the post-Riemannian framework in the manner Weyl added conformal symmetry to Einstein's gravity. In this way the gauge fields that correspond to shears and dilations become massive. In the merger setup, however, the metric tensor and the line-element are absolutely closed, and their absolute closure plays a role similar to the role played by the equivalence principle in a metric spacetime; symmetry breaking therefore becomes redundant.

It is impressing that gravity and non-Abelian YM theory are naturally interlaced into such an elegant fabric. Without supersymmetry, and without need for extra dimensions, the two physical sectors coexist as two complementary symmetry structures that make-up a single spacetime entity - the merger. The ultimate goal in this direction would probably be to extend the validity of the theory to the quantum regime, or may be to extract quantum phenomena from the merger setup itself. This task would probably require further refinements in the underlying geometrical framework. Perhaps it even requires giving-up locality in which case the whole setup should be entirely revised.

A Integration formulas and topological considerations

A.1 The 1-st Chern class and its associated charge

The 1-st Chern class associated with the frame bundle can easily be calculated using the 3-rd structural identity, $(D_{\varpi}Q)_{(\alpha\beta)} = -R_{(\alpha\beta)}$. Transvecting $(D_{\varpi}Q)_{(\alpha\beta)}$ with $g^{\alpha\beta}$, and 'smuggling' the latter into the exterior derivative of $Q_{(\alpha\beta)}$ gives:

$$g^{\alpha\beta}\left(D_{\varpi}Q\right)_{(\alpha\beta)} = d\left(g^{\alpha\beta}Q_{(\alpha\beta)}\right). \tag{135}$$

Therefore, the trace of the 3-rd identity reads

$$R^{\alpha}{}_{\alpha} = d\varpi^{\alpha}{}_{\alpha} = -dQ^{\alpha}{}_{\alpha}. \tag{136}$$

Let us now construct a $GL(n,\mathbb{R})$ frame-bundle, whose basespace has the topology of a 2-sphere S^2. The Weyl covector $Q^{\alpha}{}_{\alpha}$ on the northern and southern hemispheres is denoted, respectively, by ${}^{+}Q^{\alpha}{}_{\alpha}$ and ${}^{-}Q^{\alpha}{}_{\alpha}$. Integrating $R^{\alpha}{}_{\alpha}$ on the sphere, making use of eq. (136), and employing Stokes' theorem gives:

$$\int_{S^2} R^{\alpha}{}_{\alpha} = -\oint_{S^1} {}^{+}Q^{\alpha}{}_{\alpha} + \oint_{S^1} {}^{-}Q^{\alpha}{}_{\alpha} = 0, \tag{137}$$

because $Q^{\alpha}{}_{\alpha}$ is gauge-invariant over the whole sphere (hence ${}^{+}Q^{\alpha}{}_{\alpha} = {}^{-}Q^{\alpha}{}_{\alpha}$). In the frame geometrical picture $\varpi^{\alpha}{}_{\alpha}$ $(= -Q^{\alpha}{}_{\alpha})$ is gauge invariant since the non-linear terms generated in the transformation of $\varpi^{\beta}{}_{\alpha}$ come solely from its rotational piece.

Consider next a $GL(n,\mathbb{R}) \times U(N,\mathbb{C})$ *foliar* bundle [7] whose basespace is topologically a 2-sphere. In this case, as we have already seen, the segmental curvature takes the form:

$$\mathcal{R}^{\alpha}{}_{\alpha}(\varpi) = d\varphi^{\alpha}{}_{\alpha} + \varphi^{\alpha}{}_{\beta} \wedge \varphi^{\beta}{}_{\alpha}. \tag{138}$$

We denote the distortion 1-form on the northern and southern hemispheres by ${}^{+}\varphi$ and ${}^{-}\varphi$, respectively. Let k_{+-} be the transition function from the northern hemisphere to the southern

hemisphere for the $U(N,\mathbb{C})$ sub-bundle, and let also h_{+-} be the corresponding transition function for the $GL(n,\mathbb{R})$ sub-bundle. Making use of formula (76), the trace in frame-space (tr) of the distortion 1-form on the two hemispheres satisfies:

$$\begin{aligned} \operatorname{tr}\left({}^{+}\varphi\right) &= k_{+-}\operatorname{tr}\left[h_{+-}\left({}^{-}\varphi\right)h_{+-}^{-1}\right]k_{+-}^{-1} + nk_{+-}dk_{+-}^{-1} \\ &= k_{+-}\operatorname{tr}\left({}^{-}\varphi\right)k_{+-}^{-1} + nk_{+-}dk_{+-}^{-1}; \end{aligned} \tag{139}$$

taking the trace of (139) also in the representation space of U (Tr) yields,

$$\operatorname{Tr}\operatorname{tr}\left({}^{+}\varphi\right) = \operatorname{Tr}\operatorname{tr}\left({}^{-}\varphi\right) + n\operatorname{Tr}\left(k_{+-}dk_{+-}^{-1}\right). \tag{140}$$

Note that the term $\operatorname{tr}\left(h_{+-}dh_{+-}^{-1}\right)$ is absent in formula (139) because such a term would have emerge from the rotational piece in the connections (where the derivatives are found), and not from the distortion 1-form which transforms covariantly as a frame-space coordinate tensor.

Employing Stokes' theorem, and making use of formula (140), the generalized 1-st Chern class associated with the foliar bundle, namely $\operatorname{Tr}\operatorname{tr}\mathcal{R}$, is integrated to yield:[27]

$$\int_{S^2}\operatorname{Tr}\operatorname{tr}\mathcal{R} = n\oint_{S^1}\operatorname{Tr}\left(k_{+-}dk_{+-}^{-1}\right). \tag{141}$$

If the internal gauge field A is Abelian, the charge simply equals $2\pi mn$, $m\in\mathbb{Z}$ being the winding number associated with S^1. Otherwise, if A is non-Abelian, the charge is characterized by the mappings of the circle into domains in group-space,[28] and its magnitude is determined by the volume of these domains. For a traceless A, the charge vanishes whatsoever.

[27]Note that $\operatorname{Tr}\operatorname{tr}(\varphi\wedge\varphi)$ identically vanishes due to the fact that φ is a 1-form.

[28]$udu^{-1} = u\left[u^{-1}(\theta+d\theta) - u^{-1}(\theta)\right]\in U$, where $\theta\in\mathfrak{u}$ is the non-Abelian group angle.

A.2 Non-Abelian Stokes' theorem for the Weyl covector

Integrating formula (135) over a compact 2-domain Σ, whose boundary is $\partial\Sigma$, assuming that $g^{\alpha\beta}Q_{(\alpha\beta)}$ is nowhere singular there, yields

$$\int_{\Sigma} g^{\alpha\beta}\left(D_{\varpi}Q\right)_{(\alpha\beta)} = \oint_{\partial\Sigma} g^{\alpha\beta}Q_{(\alpha\beta)}, \tag{142}$$

which may possibly be interpreted as the non-Abelian analogue of Stokes' theorem for the non-metricity-trace (or Weyl's covector).

In an $\left(L_n\left[A\right],g\right)$, however, $Q_{(\alpha\beta)}$ no longer commutes with itself. Consequently, formula (135) should be replaced by

$$g^{\alpha\beta}\left(D_{\varpi}Q\right)_{(\alpha\beta)} = d\left(g^{\alpha\beta}Q_{(\alpha\beta)}\right) + Q^{(\alpha\beta)}\wedge Q_{(\alpha\beta)}. \tag{143}$$

Having in mind that the non-Abelian $Q_{(\alpha\beta)}$ transforms non-linearly with respect to internal gauge transformations, we should better select a 2-domain Σ that can be covered by a *single* patch over which $Q_{(\alpha\beta)}$ is globally defined. Integration now yields:

$$\int_{\Sigma}\left[g^{\alpha\beta}\left(D_{\varpi}Q\right)_{(\alpha\beta)} - Q^{(\alpha\beta)}\wedge Q_{(\alpha\beta)}\right] = \oint_{\partial\Sigma} g^{\alpha\beta}Q_{(\alpha\beta)}. \tag{144}$$

This has been shown to be equivalent to the *Abelian* Stokes' theorem for the internal gauge field A in a $\left(Y_n\left[A\right],g\right)$, see eqs. (111)-(114), section 4.2.

A.3 Integration formulas for the transvected torsion

A kind of Stokes' theorem that involves covariant exterior derivatives can be derived also for the transvected torsion, $\vartheta_\alpha\wedge T^\alpha$. This is done with the aid of the Nieh-Yan (NY) topological 4-form whose ordinary post-Riemannian format reads:

$$-\,Q_{(\alpha\beta)}\wedge\vartheta^\alpha\wedge T^\beta - g_{\alpha\beta}\left(T^\alpha\wedge T^\beta + \vartheta^\alpha\wedge\vartheta^\gamma\wedge R^{\beta}{}_{\gamma}\right) \;=\; d\left(\vartheta_\alpha\wedge T^\alpha\right); \tag{145}$$

note that the NY density is a transvection field - an invariant. For recent analytical accounts of the various NY densities, the reader is referred to [19, 20], and [8].

Taking the integral of eq. (145) over a simply-connected 4-domain enclosed by a boundary that can be covered by a single patch (for example, the 4-ball) yields:

$$\int_{\Sigma} \left[-Q_{(\alpha\beta)} \wedge \vartheta^{\alpha} \wedge T^{\beta} - g_{\alpha\beta} \left(T^{\alpha} \wedge T^{\beta} + \vartheta^{\alpha} \wedge \vartheta^{\gamma} \wedge R^{\beta}{}_{\gamma} \right) \right] = \oint_{\partial\Sigma} \vartheta_{\alpha} \wedge T^{\alpha}. \tag{146}$$

From the deformation criterion, and from the 1-st structural identity we have:

$$\begin{aligned} -Q_{(\alpha\beta)} \wedge \vartheta^{\alpha} \wedge T^{\beta} &= -Q_{\alpha\beta} \wedge \vartheta^{\alpha} \wedge T^{\beta} + T_{\alpha} \wedge T^{\alpha}, \\ \text{and} \quad \vartheta^{\alpha} \wedge \vartheta^{\beta} \wedge R_{\alpha\beta} &= \vartheta_{\alpha} \wedge \left(D_{\varpi} T \right)^{\alpha}, \end{aligned} \tag{147}$$

respectively. With the aid of these two relations, formula (146) can be rewritten in the compelling form:

$$\int_{\Sigma} \vartheta_{\alpha} \wedge \left[T^{\beta} \wedge Q^{\alpha}{}_{\beta} - \left(D_{\varpi} T \right)^{\alpha} \right] = \oint_{\partial\Sigma} \vartheta_{\alpha} \wedge T^{\alpha}, \tag{148}$$

which one may tentatively interprets as the non-Abelian generalization of Stokes' theorem for the transvected torsion $\vartheta_{\alpha} \wedge T^{\alpha}$.[29]

Otherwise, consider a $GL(n, \mathbb{R})$ frame-bundle whose basespace is topologically a 4-sphere S^4. In this case the integration of the r.h.s of eq. (145), which splits into two integrations over the boundaries of the northern and southern 4-hemispheres (namely, over the equatorial 3-sphere), vanishes due to the fact that $\vartheta_{\alpha} \wedge T^{\alpha}$ is everywhere invariant.[30] Making use of the upper equation in (147) we arrive at the relation:

$$\int_{S^4} \vartheta^{\alpha} \wedge \vartheta^{\beta} \wedge R_{\alpha\beta} = \int_{S^4} \vartheta^{\alpha} \wedge T^{\beta} \wedge Q_{\alpha\beta}. \tag{149}$$

[29] In a Weyl-Cartan spacetime, the two integrands in eq. (148) identically vanish.

[30] A similar statement was already made in [20].

We now turn to the $(L_n[A], g)$ case. Here, due to the non-commutativity of the distortion φ, one encounters an additional commutator term in formula (145):

$$-Q_{(\alpha\beta)} \wedge \vartheta^\alpha \wedge T^\beta - g_{\alpha\beta}\left(T^\alpha \wedge T^\beta + \vartheta^\alpha \wedge \vartheta^\gamma \wedge \mathcal{R}^\beta{}_\gamma\right) - \vartheta_\alpha \wedge \left[\!\left[Q^\alpha{}_\beta, T^\beta\right]\!\right] \;=\; d\left(\vartheta_\alpha \wedge T^\alpha\right); \tag{150}$$

A compelling property of the torsion in a merged spacetime is that, despite the fact that T^α transforms non-linearly with respect to internal gauge transformations, $\vartheta_\alpha \wedge T^\alpha$ transforms covariantly after all.[31] This follows directly from formula (87):

$$T^\alpha \;\mapsto\; uT^\alpha u^{-1} + udu^{-1} \wedge \vartheta^\alpha \;\Rightarrow\; \vartheta_\alpha \wedge T^\alpha \;\mapsto\; \vartheta_\alpha \wedge uT^\alpha u^{-1}. \tag{151}$$

Consider then a $GL(n,\mathbb{R}) \times U(N,\mathbb{C})$ *foliar* bundle whose basespace is taken to be the 4-sphere. Integrating the trace (Tr) of formula (150) over the sphere, making use of the fact that $\mathrm{Tr}\left(\vartheta_\alpha \wedge T^\alpha\right)$ is an invariant of the two symmetries in the foil, we get:

$$\begin{aligned}\int_{S^4} \mathrm{Tr}\left(\vartheta^\alpha \wedge \vartheta^\beta \wedge \mathcal{R}_{\alpha\beta}\right) &= \int_{S^4} \mathrm{Tr}\left(-Q_{\alpha\beta} \wedge \vartheta^\alpha \wedge T^\beta - \vartheta_\alpha \wedge \left[\!\left[Q^\alpha{}_\beta, T^\beta\right]\!\right]\right) \\ &= \int_{S^4} \mathrm{Tr}\left(\vartheta^\alpha \wedge T^\beta \wedge Q_{\alpha\beta}\right);\end{aligned} \tag{152}$$

Hence, formula (152) generalizes formula (149) to the case of $(L_n[A], g)$.[32] Finally, one may easily verify that the non-Abelian extension of Stokes' theorem for the transvected torsion in an (L_n, g), as given by eq. (148), holds in this form also in an $(L_n[A], g)$.

[31] This property is shared also by the off-diagonal elements of $\varphi^\beta{}_\alpha$.

[32] Making use of the transformation rules for $Q_{\alpha\beta}$ (eq. (77)) and T^α (eq. (87)), the reader may verify that $\mathrm{Tr}\left(\vartheta^\alpha \wedge T^\beta \wedge Q_{\alpha\beta}\right)$ is indeed (internally) gauge-invariant, as eq. (152) requires.

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